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División de Ciencias Básicas

"Dinámica cuántica de una partícula cargada inmersa en un campo electromagnético estático y las consecuencias de una nueva solución"

Tesis Profesional

que para obtener el grado de

Doctor en ciencias en física

presenta

M. en C. Jorge Alfonso Lizárraga Brito

director de tesis

Dr. Gustavo López Velázquez

GUADALAJARA, JALISCO. ENERO 2024

CUCEI/SAC/CDDCF/017/2023

DRA. MARIA TERESA ROMERO GUTIERREZ
JEFA DE LA UNIDAD DE POSGRADO
CUCEI
P R E S E N T E

Por este medio me permito informar a Usted que la Junta Académica del Doctorado en Ciencias en Física determinó por unanimidad como **ACEPTADO** el Protocolo de Tesis para obtener el grado de Doctor en Ciencias en Física al M. en C. Jorge Alfonso Lizárraga Brito la cuál llevará por nombre:

"Dinámica cuántica de una partícula cargada inmersa en un campo electromagnético estático y las consecuencias de una nueva solución"

Elaborado por el M. en C. Jorge Alfonso Lizárraga Brito para obtener el grado de **Doctor** en Ciencias en Física la consideramos apta para su **impresión en el idioma inglés y presentación**.

Se anexa copia.

Sin más de momento, agradezco de antemano su colaboración.

A T E N T A M E N T E
Guadalajara, Jalisco 22 de noviembre del 2023
"2023, Año del fomento a la formación integral
Con una Red de Centros y Sistemas Multitemáticos"
MIEMBROS DE LA JUNTA ACADÉMICA DEL DOCTORADO EN CS. EN FÍSICA

Dr. Isabel Sainz Abascal **(Presidenta)**

Dr. Andrei Klimov

Dr. Armando Pérez Centeno

Dra. Claudia Moreno González

Dr. Gerardo Ramos Larios

Guadalajara, Jalisco a 22 de noviembre del 2023

DRA. MARIA TERESA ROMERO GUTIERREZ
JEFA DE LA UNIDAD DE POSGRADOS
CUCEI
PRESENTE

Por este conducto hacemos de su conocimiento que hemos revisado el trabajo de tesis **"Dinámica Cuántica de una partícula cargada inmersa en un campo electromagnético estático y las consecuencias de una nueva solución"** elaborado por el M. en C. **Jorge Alfonso Lizárraga Brito** para optar al grado de Doctor en Ciencias en Física.

Nos es grato comunicarle que encontramos este trabajo listo para su impresión y posterior defensa.

Sin más por el momento, quedamos a sus órdenes para cualquier aclaración.

ATENTAMENTE

G. López

Dr. Gustavo López Velázquez
Director de Tesis

Dr. Alexander Nesterov
lector revisor

Dr. José Luis Romero Ibarra
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“Quantum Dynamics For Single Charged Particle With Static Electromagnetic Field and the Consequences of a New Solution”

Professional Thesis

to obtain the title of

Doctor Of Philosophy In Science Of Physics

presents

M. S. Jorge Alfonso Lizárraga Brito

thesis director

Dr. Gustavo López Velazquez

GUADALAJARA, JALISCO. JANUARY 2024

Contents

Abstract

En esta tesis se presentan los resultados obtenidos de analizar el Hamiltoniano no relativista para campos electromagnéticos constantes. El análisis se realizó dandole especial prioridad a la coherencia matemática de las ecuaciones diferenciales parciales que se obtienen seleccionando las diferentes normas para describir la dirección de los campos magnéticos. Aunado a ello, el análisis mostrado pretende brindar una perspectiva diferente a la manera de trabajar con estos sistemas abordando las ecuaciones como ecuaciones no separables en todas sus coordenadas y el hecho de que un operador conservado puede actuar como generador de soluciones a dichas ecuaciones las cuales no necesariamente son proporcionales a la función de onda original. Finalmente, para el caso en que un campo electromagnético es aplicado, se hace la observación de cómo estos resultados implicarían la cuantización de la resistividad, hecho ya observado experimentalmente en fenómenos como el efecto de Hall fraccionario.

Abstract

In this thesis presents the results obtained from analyzing the non-relativistic Hamiltonian for constant electromagnetic fields. The analysis was carried out giving special priority to the mathematical coherence of the partial differential equations that are obtained by selecting the different gauges to describe the direction of the magnetic fields. In addition to it, the analysis shown aims to provide a different perspective on the way of working with these systems, addressing the equations as non-separable equations in all their coordinates and the fact that a conserved operator can act as a generator of solutions of the equations. Finally, for the case in which an electromagnetic field is applied, tit is shown how these results implies the quantization of resistivity, a fact already observed experimentally in phenomena such as the fractional Hall effect.

Chapter 1

Introduction

The problem of the non relativistic quantum dynamics of a charged particle moving on a surface under an electromagnetic field, where the magnetic field is perpendicular to its motion, was studied many years ago by Landau [1]. His solution is well known, and the spectrum of the system is known as Landau's levels. This solution has been used by many researchers to try to understand the physical phenomenon called Quantum Hall Effect [2–5], which in turns is useful to understand what is called Topological Insulators. For instance, in 1982 Kohmoto, Nightingale, and den Nijs showed that the Hall conductivity, using the Kubo's formula, can be expressed as the Chern number which is a topological invariant [6], study that was latter extended by Avron, Seiler and Simon showing that those were the only topological invariant possible [7] and they also gave a geometrical interpretation [8]. Another example is the work of Kohmoto, which inquired in the topological aspects of a two dimensional wave function for electrons, showing that the quantization of the Hall conductivity is related to the number of zeros of wavefunctions in the magnetic Brillouin zone [9]. For the constant magnetic field B and the vector potential chosen as $\mathbf{A} = (-By, 0, 0)$, so called Landau's gauge, Landau's solution of the Shrödinger equation is based on the fact that component $\hat{p}_x$ of the generalized linear momentum commutes with the Hamiltonian, which leads him to propose a solution having all the variables separated, that is, the eigenfunctions of the Hamiltonian is written as the product of functions where each function depends on just a single variable. However, when one looks carefully the partial differential equation defined by the problem, one realized that this partial differential equation can admit a non-separable variable solution. In this work we will find a non-separable solution for this problem and will explore the possible consequences of this solution, focusing on seeing something analogous to Quantum Hall Effect.

But first lets see a brief history of the quantum Hall effect. In 1879 Edwin Herbert Hall (1855-1938) while working on his doctoral thesis in Physics under the supervision of Henry Augustus Rowland, performed an experiment consisted of exposing thin gold leaf on a glass plate and tapping off the gold leaf at points down its length and applying a magnetic field perpendicular to the sample. The effect is a potential difference (Hall voltage) on opposite sides of a thin sheet of the material through which an electric current is flowing. This transverse current is known as Hall current which generates the known Hall conductivity and is usually denoted by σ_H [10]. Therefore, the encyclopedic definition of the this phenomena, now called Hall effect, can be read as:

The Hall effect is the production of a potential difference (the Hall voltage) across an electrical conductor that is transverse to an electric current in the conductor and to an applied magnetic field perpendicular to the current.

On the other hand, one could think about finding the quantum side of this phenomena where the Hall conductivity, or resistivity (the inverse of the conductivity), is quantized. One of the first attempts to find this quantum phenomena was made by Klaus von Klitzing presented in his 1974 work [11] where he showed his measurements results obtained in a p-type and n-type channel silicon metal-oxide-semiconductor field-effect transistor (MOSFET). However, what is curious about it is that Klitzing thought that the n-type MOSFET results he got were the same than those presented by Fowler, Fang, Howard, and Stiles [12, 13]. That being said, it turns out that actually Klitzing results were different from those already given by the other authors and what is more impressing is that they were the first measurement of the quantization of Halls resistivity, unfortunately, those results were not published [14].

Afterwards, in 1978, Jun-ichi Wakabayashi and Shinji Kawaji published they experimental results in the Journal of the phsical society of japan [15]. In their experiment they used a long rectangular sample of n-type MOS inversion layers on Si(100) surfaces applying a strong magnetic field of 100(kOe) at a temperature of 1.6(K). Besides performing the experiment, they went further by comparing the experimental data with the one predicted by the theory and is curious that they reported a mismatch of the data given by the theory. Despite that they gave a couple of possible explanation trying to justify this mismatch, there is still another observation that they did not realized about. The theoretical prediction was made using the two dimensional theory of Quantum transport developed by Tsuneya Ando and Yasutada Uemura which was pubished in 1974 divided among 4 papers [16–19], however, in the reference [16] one can read that the solution used to developed the whole theory is written as

$$\phi_{NX}(x, y) = \exp\left(i\frac{xy}{2l^2} - i\frac{Xy}{l^2}\right)\phi_N(x - X) \tag{1.1}$$

where N and X are the quantization index, $l^2 = c\hbar/qB$, ϕ_N is the harmonic oscillator solution, eq.(2.44), and they claimed that this is the solution of the Schrödinger's equation using the symmetric gauge eq.(2.120). Nevertheless is easy to realized that they obtained this solution using two facts. First, the solved Schrödinger's equation using Landau's gauge eq.(2.18) and applying Landau's solution too [1] which we already know that has the pathology of being a separable variable solution for a non separable variable partial differential equation. Second they applied the argument of gauge invariance which basically states that being the symmetric gauge denoted as $\mathbf{A}_S$ and the one describing Landau's gauge denoted as $\mathbf{A}_L$ they are related by the equality

$$\mathbf{A}_S = \mathbf{A}_L + \nabla S, \tag{1.2}$$

where S is a scalar function and the solution are related by a unitary transform written as

$$\Psi_S = e^{i\frac{q}{\hbar c}S}\Psi_L, \tag{1.3}$$

where Ψ_S and Ψ_L is the solution for the symmetric gauge and Landau's gauge respectively.

Around a couple of years later, in 1980, Klitzing struck again, along with Dorda and Pepper they presented their new experimental results carried out in a Si (100) n-channel field-effect transistor [3]. This time, Klitzing decided to polish his previous experiment merging it with its photoconductivity experiment [20] which creates phosphorous impurities in silicon, they achieved to measured the Hall voltage as a function of the MOSFETs voltage gate showing that the Hall conductivity where quantized in multiples of the electron charge, q, squared divided by Plank's constant, h, that is

$$\sigma_H = k\frac{q^2}{h}, \quad k \in \mathbb{Z}^+, \tag{1.4}$$

or in therms of the resistivity, which is the inverse of the conductivity,

$$\rho_H = \frac{1}{k}\frac{h}{q^2}, \tag{1.5}$$

whereas the longitudinal resistivity drops down to zero, $\rho_L = 0$. Nowadays the constant which quantize the Hall resistivity is called the *Kltizing's constant* [21]

$$\frac{h}{q^2} = 25812.80745(\Omega), \tag{1.6}$$

and this phenomena is now called *Integer Quantum Hall effect* (IQHE).

Later on, in October 1981 Dan Tsui and Horst L. Stormer begun to work in a low electron-density GaAs/AlGaAs sample ($n = 1.23 \times 10^{11} cm^{-2}$) with an exceedingly high mobility of $\mu = 90000 cm^2/V sec.$ where Hall measurement where performed at the temperature of liquid He ($4.2K$). Although, this time they measured the resistivity as a function of the magnetic fields intensity, they were able to replicates the IQHE for a magnetic fields range of $0 < B < 51(kG)$ but as they kept rising this quantity up to three times more, that is $B \sim 150(kG)$, they observed that while the longitudinal resistivity vanished, the Hall resistivity begun to quantized as three times the Klitzing's constant, that is

$$\rho_H = 3\frac{h}{q^2}, \tag{1.7}$$

or in terms of the conductivity

$$\sigma_H = \frac{1}{3}\frac{q^2}{h}. \tag{1.8}$$

This unexpected result where publish in Physical Review Letters in March 1982 [22] and with it the phenomena denominated as *Fraction Quantum Hall effect* (FQHE) was born [23, 24]. As the experimentalist continued to developed more precises ways to measure this Hall resistivity along with the devices improvements, it became clearer that the quantum Hall effect has two main characteristics, the first one is that the Longitudinal resistivity drops to zero and the second one is that the Hall resistivity is quantized in rational multiples of the Klitzing's constant as [25–31]

$$\rho_H = \frac{l}{k}\frac{h}{q^2}, \quad l, k \in \mathbb{Z}^+. \tag{1.9}$$

Is remarkable that the latter improvements of this phenomena has achieved an accuracy up to few parts per billion [32, 33].

Meanwhile, the experimentalist were astonished for this new results, the theoretical scientist were working in a hypothesis capable to explain the main characteristics of this effect. The first attempt to understand the IQHE relies on a mix of the classic and quantum theory. Basically we can summarize the idea as follows: considere a gas of charged particles moving with a common velocity $v\hat{i}$, this corresponds to a current density $j_x = qnv$ where q is the particles charge and n is the density per unit area in the sample. Then, the Lorentz force that is to be balanced by the electric force $qE_y\hat{k}$ is given by $qvB\hat{k}$. Therefore $E_y = vB$ and the current density satisfied

$$j_x = -qn\frac{E_y}{B}. \tag{1.10}$$

This equation suggest the definition of the Hall resistivity as [34]

$$\rho_H = \frac{E_y}{j_x} = \frac{B}{qn}. \tag{1.11}$$

Then, the quantum realm hops in as follows. In 1977 Landau's book was published, in it an attempt to solve Hamiltonian (2.19) via his *ansatz* [1] can be found, where he proposed free particle dynamic in one of the directions of the particle and then he proposed periodicity in that same direction, lets say $\psi(x, y + L_y) = \psi(x, y)$ and by analysis of the maximum and minimum values of the center of the oscillations he concluded that the next quantity must be quantized [1, 34]

$$\frac{m\omega_c}{\hbar}A = 2\pi l, \quad l \in \mathbb{Z}. \tag{1.12}$$

where m is the particle mass, $\hbar$ is the Plank's constant, $A = L_xL_y$ is the area of the two dimensional system with lengths L_x and L_y and ω_c is the same cyclotron frequency defined in section (2.1) which can be written

[1] Lets remember to the reader that the dictionary defines the word ansatz as an assumption about the form of an unknown function which is made in order to facilitate solution of an equation or other problem.

in MKS units as $\omega_c = qB/m$, then, the above equality tell us that necessarily the *magnetic flux* must be quantized

$$BA = 2\pi l \frac{\hbar}{q}. \tag{1.13}$$

Finally, the numbers of states per unit area is the same as the electron density due to Pauli exclusion principle [35, 36] so it can be written as

$$n = \frac{qB}{h}, \tag{1.14}$$

where we have used the definition of the Plank's constant as $h = 2\pi\hbar$. Substituting this result in the classic definition of Hall resistivity (1.11) we have that

$$\rho_H = \frac{h}{q^2}, \tag{1.15}$$

which correspond to the IQHE with quantization index of $k = 1$ and $l = 1$. Finally, using an argument of a density increment they manage to write the equation (1.5) [34]. Alternatively, Robert B. Laughlin used a gauge invariant argument to try to explain this same quantization [4], further works on this argument were developed in references [37–41].
The influence of the impurities in the phenomena measurements were analyzed mainly by Prange, which modeled them as a Dirac's delta potential [5]. His main result is that due to the imperfections of the material, there exist localized states in the sample which do not contribute to the final Hall resistance measurements, however, somehow the non localized states which travels nears the localized ones manage to have a higher contribution which compensates the no contribution of the localized particles. Again, alternatively, Laughlin struck again with a gauge invariance argument to analyze the effect of the impurities [42].

Laughlin continued his work trying to explain the FQHE using his many-body wave function hypothesis, what is now called Laughlin's wave function, but before we present it to the reader is necessary to inquiry a little bit more in his work to understand the pathologies it has [43–46]. His start point consisted in dealing with three two dimensional particles with Coulomb interaction [47], he took advantage of the fact that the angular momentum eq.(2.130) commutes with any scalar function that depends only of the radius, that is

$$[\hat{L}_z, f(r)] = 0, \tag{1.16}$$

and knowing that using the symmetric gauge this commutation arguments holds for the whole Hamiltonian considering the interaction potential, he managed to use Landau's *ansatz* [1] to write his solution [43] for the two and three body problems, the details of this solution can be seen in reference [48]. Then, his approach was to deal with the interaction potential as if it was a perturbation small enough to apply perturbation theory [49] which used it to explain FQHE found by Störmer with quantum index $k = 1$ and $l = 3$ in eq.(1.7). Afterwards, Laughlin tried to generalized his hypothesis to a N-body systems by means of Jastrow's *ansatz*[2] [50] and even more he used his own *ansatz* to write down his wave function as [44, 51]

$$\psi(z_1, ..., z_N) = \prod_{i<j} (z_i - z_j)^m \exp\left(\sum_i \frac{|z_i|^2}{4l_o^2}\right), \tag{1.17}$$

where l_0 is a constant and z_i are the complex coordinates of the particles. This is the trial N-body ground wave function, that is, for the lowest landau level which Laughlin claimed that can not be degenerated [44], however, as we have already seen in section (2.2.1) the fact that the angular momentum is conserved implies that the solution of the Schrödinger's equation is actually numerable degenerated, more over, in 1984 Tao and Wu have used Laughlin's gauge invariance own argument [4] to prove that the ground state must be degenerated[3] [52].

[2] This is contradictory to his previous work since this *ansatz* applies only for strong interaction forces contrary to perturbation theory which applies to small ones.

[3] This lack of coherence is due to the assumption that the conservation of an operator implies that it shares basis with the Hamiltonian but, as we have shown through this thesis work, that is not true.

Further research about the ground state degenerancy were made by Haldene and Rezayi [53] and Su [54] respectively. One the other hand, this trial wave function allowed Laughlin to explain FQHE only for Hall conductivities equal to

$$\sigma_H = \frac{1}{k}\frac{q^2}{h}, \tag{1.18}$$

where k is an odd integer. However, one needs to remark that this phenomena is not constrained to only odd denominators of the conductivity [25, 30, 55]. Besides, the physical interpretation of this phenomenon using this approach is quite counterintuitive as has been notice by Störmer [23]. Even though if Laughlin's hypothesis has its basis in a several body approach, it turns out that its principal conclusion is that the electrons gather together to build up a bigger *quasiparticle*[4] which has a total charge of a fraction of one of the individuals electrons that forms it. To have a better picture of this situation, imagine that three electrons gather together to form a *quasiparticle* of total charge e^*, despite that each electron has an individual charge of $e = 1.60217663 \times 10^{-19}(C)$ it turns out that, for Laughlin, the total charge of the *quasiparticle* is $e^* = e/3$ and therefore the conductivity is a fraction of the inverse of Klitzing's constant [44]. A generalization of Laughlin's trial wave function was made by Jain who constructed a more general trial wave function, however, the methods used to deduced it are pretty similar to those used by Laughlin, where they only guess a solution for the Hamiltonian setting the mathematical formalism aside [56–59]. As a continuation of his research, Laughlin used his ground state trial wave function (1.17) to prove that the resonating-valence-bond and the fractional quantum Hall effect states are (in his own words) the same thing, which turns out to be posible due to the creation of *quasiparticles* obeying fractional statistics, often called *anyons* [5], which lead to a new superconductivity hypothesis having its basis in the FQHE [60–63]. However, this *anyonic* based hypothesis was refuted by means of muon-spin relaxation (μSR) experiment where the magnetic field produced by the *quasiparticle* (and predicted by the theory) should be of $\sim 30(G)$ but they detected a magnetic field of around 10% that amount, besides, that measured magnetic field couldn't be associated to the creation of *anyons*, instead, it was due to the anisotropic nature of the muon-nuclear dipolar-interaction [64]. Further different experimental developments has confirmed this results where they could not detect the existence of a *quasiparticle* in their respective set up [65–73]. On the other hand, there are also experimentalist who claim to have detected a fractional charged quasiparticle [74–76]. Anyway, we must say that FQHE is still considered an open research problem [77].

To sum up, the quantum Hall effect has two branches, the integer quantum Hall effect, which is characterized for having a quantized resistivity of the form eq.(1.9) with $l = 1$, $k \in \mathbb{Z}^+$ and the fractional quantum Hall effect, which is characterized for having a quantized resistivity of the form eq.(1.9) with $l \in \mathbb{Z}^+$ and $k = 1$. In general, is also possible to find a mixed phenomena of this two branches having a resistivity as shown in eq.(1.9). This phenomena appears when a perpendicular electromagnetic field is applied to a two dimensional electron gas and its temperature is reduced at the point that the systems is in the lowest Landau level. The integer part is explained as a single particle phenomena, while the fractional part is explained as a several particle phenomena. Finally, in order to have a new perspective of this phenomena, is mandatory to forget the old approach that has a lack of mathematical formalisms attached to it coming from the unmeasured use of *ansatzes* used to solve the respective partial differential equations.

Throughout all this work we have shown how to work mathematically with the Schrödinger equation that describes the non-relativistic quantum dynamics of a charged particle under the influence of an electromagnetic field. However, we have not inquiry about the physical implications about the obtained solutions yet, that is the main objetive of this section. We will show that a similar phenomenon like the one called *Quantum Hall effect*, which appears, experimentally, by the presence of the electromagnetic field in a two dimensional material at low temperatures and its main characteristic is the resistivity quantization [3], can be obtained

[4]Etymology: "quasi" and "particle", used to refer to any entity that has some characteristics of a distinct particle, but comprises a grouping of multiple particles.

[5]Etymology: "any" and "on", used to refer to an elementary particle or particle-like excitation having properties intermediate between those of bosons and fermions.

form the solutions found. Nevertheless, we must mention that there exist a couple of cases where this phenomenon is generated but the physical reasons that gives rise to them are different from each others, for instance, the *Quantum Anomalous Hall effect* where the magnetic field is absented, that is $\mathbf{B} = 0$, but the magnetization of the material is present and the spin-orbit coupling of the particles has influence on the dynamics of the particles producing a Quantum Hall effect like phenomena [78, 79] and the *Quantum Spin Hall effect* that does not require the application of a large magnetic field, non magnetization of the material but still have a quantized spin-Hall conductance [80]. However, this last two phenomenon are not worked out through this thesis.

The construction of the present thesis is as follows. In Chapter (2) we give a brief review of the Hamiltonian we are going to work with, which is the non-relativistic system under the effect of a electromagnetic field. In section (2.1) we work with the system under the effect of a magnetic field only and such that the field is described by the Landau's gauge. After finding a non-separable variable solution, we continue in section (2.1.1) by analyzing the degeneration of the system which is given by the application of the conserved operators. Later on, in section (2.2) we proceed by analyzing the same system but using the so called symmetric gauge, which, even though if classically they describe the same system, it turns out that Schrödinger's equation is different from that one described using Landau's gauge. Followed by the analysis of the degeneration of this system in section (2.2.1). After settle down the methods to work with this kind of equations, we move on to the problem where an electromagnetic field is present, this is worked out in section (2.3) and its degeneration is presented in section (2.3.1). It is important to mention that all the results are accompanied by their rigorous mathematical details which can be find in the appendices (A,B,C) respectively. Finally, in chapter (3) we use the obtained results to calculate the current produced by the particle and explain how could this result be associated to a quantum Hall like phenomena. However, this last chapter is strongly based on the results obtained on the appendix (D), therefore is recommended to check it before approaching to this last chapter.

Chapter 2

Hamiltonian for a single charged particle

It is well known that the non relativistic dynamics of an electric charge motion with the electromagnetic field is given by the Lorentz' force

$$\frac{d(m\mathbf{v})}{dt} = q\mathbf{E} + \frac{q}{c}\mathbf{v} \times \mathbf{B}, \tag{2.1}$$

where c is the speed of light, m and q are the mass and the charged of the particle, and $\mathbf{E}$ and $\mathbf{B}$ are the electric and the magnetic fields. These fields are given in terms of the scalar and vector potential Φ and $\mathbf{A}$ as

$$\mathbf{E} = -\nabla\Phi - \frac{q}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{2.2}$$

It is known [81] that the dynamics of the charged particle motion is written in terms of these potential as

$$\frac{d}{dt}\left(m\mathbf{v} + \frac{q}{c}\mathbf{A}\right) = -\nabla\left(q\Phi - \frac{q}{c}(\mathbf{v}\cdot\mathbf{A})\right). \tag{2.3}$$

The canonical momentum and the potential are

$$\mathbf{p} = m\mathbf{v} + \frac{q}{c}\mathbf{A}, \tag{2.4}$$

and

$$U = q\Phi - \frac{q}{c}(\mathbf{v}\cdot\mathbf{A}), \tag{2.5}$$

is the *generalized potential energy* [81], and one can calculate a Hamiltonian for the system as

$$\hat{\mathbf{H}} = \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{q}{c}\mathbf{A}\right)^2 + V \tag{2.6}$$

where $V = U + \frac{q}{c}(\mathbf{v}\cdot\mathbf{A})$. Now, the evolution of an operator $\hat{f}$ in Heisenberg scheme is given by the following equation

$$\frac{d\hat{f}}{dt} = \frac{1}{i\hbar}[\hat{f}, \hat{H}] + \frac{\partial \hat{f}}{\partial t}. \tag{2.7}$$

Using the Hamiltonian (2.6)

$$\hat{\mathbf{H}} = \frac{1}{2m}\left(\left(\hat{p}_x - \frac{q}{c}A_x\right)^2 + \left(\hat{p}_y - \frac{q}{c}A_y\right)^2 + \left(\hat{p}_z - \frac{q}{c}A_z\right)^2\right) + V, \tag{2.8}$$

and the following relation for any operators A, B, and C given by $[AB, C] = A[B, C] + [A, C]B$, we see that

$$\left[\hat{x}, \left(\hat{p}_x - \frac{q}{c}A_x\right)^2\right] = 2i\hbar\left(\hat{p}_x - \frac{q}{c}A_x\right), \tag{2.9}$$

$$\left[\hat{x}, \left(\hat{p}_y - \frac{q}{c}A_y\right)^2\right] = 0, \tag{2.10}$$

$$\left[\hat{x}, \left(\hat{p}_z - \frac{q}{c}A_z\right)^2\right] = 0, \tag{2.11}$$

$$[\hat{x}, V] = 0, \tag{2.12}$$

then, using (2.7), we have that

$$\frac{d\hat{x}}{dt} = \frac{1}{m}\left(\hat{p}_x - \frac{q}{c}A_x\right), \quad \frac{d\hat{y}}{dt} = \frac{1}{m}\left(\hat{p}_y - \frac{q}{c}A_y\right), \quad \text{and} \quad \frac{d\hat{z}}{dt} = \frac{1}{m}\left(\hat{p}_z - \frac{q}{c}A_z\right). \tag{2.13}$$

Hence, the modified momentum operators can be written as the above velocity operators times the mass, that is

$$\hat{\pi}_x = \hat{p}_x - \frac{q}{c}A_x, \tag{2.14}$$

$$\hat{\pi}_y = \hat{p}_y - \frac{q}{c}A_y, \tag{2.15}$$

$$\hat{\pi}_z = \hat{p}_z - \frac{q}{c}A_z. \tag{2.16}$$

These known expressions will be useful to analyzing the degeneration of the system later on.

2.1 Hamiltonian with Landau gauge

In this section, we will star our research by focusing on the solutions without electric field and static magnetic field, that is $V = 0$, and for different gauge selection.

$$V = 0, \qquad \text{and} \qquad \mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}). \tag{2.17}$$

Let's begin analyzing the case when the magnetic field has constant magnitude, B, and pointing along z direction positively, that is, $\mathbf{B} = B\hat{k}$. One possible selection of gauge describing a field of this characteristics is the so called *Landau's* gauge which is written as follows

$$\mathbf{A} = B(-y, 0, 0), \tag{2.18}$$

then the Hamiltonian (2.8) can be written as

$$\hat{\mathbf{H}} = \frac{1}{2m}\left(\left(\hat{p}_x + \frac{qB}{c}y\right)^2 + \hat{p}_y^2 + \hat{p}_z^2\right). \tag{2.19}$$

Now, by using the definition of the *cyclotron frequency* $\omega_c = qB/mc$ and rearranging terms, we get

$$\hat{\mathbf{H}} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 + m\omega_c(\hat{p}_x y + y\hat{p}_x) + m^2\omega_c^2 y^2\right), \tag{2.20}$$

which, due to the fact that $[\hat{p}_x, y] = 0$, this Hamiltonian is written as

$$\hat{\mathbf{H}} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 + 2m\omega_c y\hat{p}_x + m^2\omega_c^2 y^2\right). \tag{2.21}$$

Now, the time independence of this Hamiltonian allow us to separate the time dependence part from the spatial dependence in the Scrödinger's equation,

$$i\hbar\frac{\partial \Psi}{\partial t} = \hat{\mathbf{H}}\Psi, \tag{2.22}$$

where $\Psi(x, y, z, t)$ is the wave function. This separation is getting by choosing this wave function of the form $\Psi(x, y, z, t) = \psi(x, y, z)e^{-i\frac{E}{\hbar}t}$ and reducing the problem to and the following eigenvalue problem

$$\hat{\mathbf{H}}\psi = E\psi, \tag{2.23}$$

or

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + i2\frac{m\omega_c}{\hbar}y\frac{\partial\psi}{\partial x} - \frac{m^2\omega_c^2}{\hbar^2}y^2\psi\right) - \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial z^2} = E\psi, \tag{2.24}$$

where we have use explicitly the operators $\hat{p}_j = -i\hbar\partial/\partial_j$ with $j = x, y, z$. A careful examination of this partial differential equation shows us that the "z" direction is the only spatial coordinate separable. In fact, by proposing a solution of (2.24) of the form

$$\psi(x, y, z) = \varphi(x, y)g(z), \tag{2.25}$$

and dividing by the same solution, one gets

$$-\frac{\hbar^2}{2m}\left(\frac{1}{\varphi}\frac{\partial^2\varphi}{\partial x^2} + \frac{1}{\varphi}\frac{\partial^2\varphi}{\partial y^2} + \frac{1}{\varphi}i2\frac{m\omega_c}{\hbar}y\frac{\partial\varphi}{\partial x} - \frac{m^2\omega_c^2}{\hbar^2}y^2\right) - \frac{\hbar^2}{2m}\frac{1}{g}\frac{\partial^2 g}{\partial z^2} = E \tag{2.26}$$

which brings about the equations

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+i2\frac{m\omega_c}{\hbar}y\frac{\partial\varphi}{\partial x}-\frac{m^2\omega_c^2}{\hbar^2}y^2\varphi\right)=E_1\varphi, \tag{2.27}$$

and

$$-\frac{\hbar^2}{2m}\frac{\partial^2 g}{\partial z^2}=E_2 g, \tag{2.28}$$

being $E=E_1+E_2$. The solution for the equation (2.28) is straightforward

$$g(z)=e^{i\frac{\sqrt{2mE_2}}{\hbar}z}. \tag{2.29}$$

Therefore, we will look for a way to solve this equation, and one very useful method is the Fourier transformation. For this case we will use the *Fourier* transform acting over the variable x, that is

$$\overline{\varphi}(\kappa,y)=\mathcal{F}_x(\phi)=\frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}}e^{i\kappa x}\phi(x,y)dx, \tag{2.30}$$

applying it to equation (2.27)

$$-\frac{\hbar^2}{2m}\left(\mathcal{F}_x\left(\frac{\partial^2\varphi}{\partial x^2}\right)+\mathcal{F}_x\left(\frac{\partial^2\varphi}{\partial y^2}\right)+i2\frac{m\omega_c}{\hbar}y\mathcal{F}_x\left(\frac{\partial\varphi}{\partial x}\right)-\frac{m^2\omega_c^2}{\hbar^2}y^2\mathcal{F}_x\left(\varphi\right)\right)=E_1\mathcal{F}_x(\varphi), \tag{2.31}$$

and using the identity (A.4) and rearranging we can write

$$\frac{\partial^2\overline{\varphi}}{\partial y^2}-\frac{m^2\omega_c^2}{\hbar^2}y^2\overline{\varphi}-\kappa^2\overline{\varphi}+2\frac{m\omega_c}{\hbar}y\kappa\overline{\varphi}=-\frac{2mE_1}{\hbar^2}\overline{\varphi}, \tag{2.32}$$

then, we can complete the squared binomial as follows

$$\frac{\partial^2\overline{\varphi}}{\partial y^2}-\frac{m^2\omega_c^2}{\hbar^2}\left(y-\frac{\hbar}{m\omega_c}\kappa\right)^2\overline{\varphi}=-\frac{2mE_1}{\hbar^2}\overline{\varphi}, \tag{2.33}$$

of course the above equation is nothing else than a displaced harmonic oscillator in the Fourier space (κ,y). We continue by making the change of variable

$$\xi=\sqrt{\frac{m\omega_c}{\hbar}}\left(y-\frac{\hbar}{m\omega_c}\kappa\right), \tag{2.34}$$

then, the differential operator changes as

$$\frac{\partial^2}{\partial y^2}=\frac{m\omega_c}{\hbar}\frac{\partial^2}{\partial\xi^2}, \tag{2.35}$$

and the equations takes the next form

$$\frac{\partial^2\overline{\varphi}}{\partial\xi^2} - \xi^2\overline{\varphi} = -\frac{2E_1}{\hbar\omega_c}\overline{\varphi}, \tag{2.36}$$

and has the solution of

$$\overline{\varphi}_n(\xi) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\xi^2}{2}\right)H_n(\xi), \tag{2.37}$$

where H_n are the Hermite polynomial and eigenvalues of

$$E_1 = E_n = \hbar\omega_c\left(n+\frac{1}{2}\right). \tag{2.38}$$

Since our original equation was in the real space (x, y), is necessary to perform the inverse of *Fourier* transform of the above expression defined as

$$\mathcal{F}_\kappa^{-1}(\overline{\varphi}_n(\xi)) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\kappa x}\overline{\varphi}_n(\xi)d\kappa, \tag{2.39}$$

then from the expression (2.34)

$$\kappa = \frac{m\omega_c}{\hbar}y - \sqrt{\frac{m\omega_c}{\hbar}}\xi, \tag{2.40}$$

$$d\kappa = -\sqrt{\frac{m\omega_c}{\hbar}}d\xi, \tag{2.41}$$

thus, the solution in the real space is $\varphi(x, y) = \mathcal{F}_\kappa^{-1}(\overline{\varphi}(\xi))$, hence, we can write

$$\varphi_n(x,y) = -\sqrt{\frac{m\omega_c}{\hbar}}\exp\left(-i\frac{m\omega_c}{\hbar}xy\right)\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp\left(i\sqrt{\frac{m\omega_c}{\hbar}}x\xi\right)\overline{\varphi}_n(\xi)d\xi, \tag{2.42}$$

using the result from appendix (A.1) equation (A.16) that shows that the Fourier transform of a harmonic oscillator is another harmonic oscillator we can write down the solution in the real space as

$$\varphi_n(x,y) = -\sqrt{\frac{m\omega_c}{\hbar}}\exp\left(-i\frac{m\omega_c}{\hbar}xy\right)\phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}x\right), \tag{2.43}$$

where it was define the function

$$\phi_n(\chi) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega_c}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\chi^2}{2}\right)H_n(\chi). \tag{2.44}$$

Then we can write the eigenvalue solution as

$$\psi_n(x,y,z) = A\exp\left(-i\frac{m\omega_c}{\hbar}xy + i\frac{\sqrt{2mE_2}}{\hbar}z\right)\phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}x\right), \tag{2.45}$$

where A is a normalization constant. However, observe that the the function ϕ_n is already normalized, that is

$$\int\limits_{-\infty}^{\infty}\left|\phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}x\right)\right|^2 dx = 1, \tag{2.46}$$

then the normalization condition is written as follows,

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}|\psi_n(x,y,z)|^2 dxdydz = 1, \tag{2.47}$$

anyway, since the variables y and z appears only in the phase, the integration over all the space will give us an indetermination, therefore we use the Born normalization [49] for this variables, that is, we assume that the particle is in a finite box of lengths (L_x, L_y, L_z) but with the condition that the length L_x is large enough such that can be considered infinity, then the normalization condition is as follows

$$\int\limits_{-L_z/2}^{L_z/2}\int\limits_{-L_y/2}^{L_y/2}\int\limits_{-\infty}^{\infty}|\psi_n(x,y,z)|^2 dxdydz = 1, \tag{2.48}$$

then this implies that $A = 1/\sqrt{L_yL_z}$ and the eigenfunctions are written as

$$\psi_n(x,y,z) = \frac{1}{\sqrt{L_yL_z}}\exp\left(-i\frac{m\omega_c}{\hbar}xy + i\frac{\sqrt{2mE_2}}{\hbar}z\right)\phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}x\right), \tag{2.49}$$

with eigenvalues of

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right) + E_2. \tag{2.50}$$

Finally, we must note that the energies are continuos in the z direction. However, we can assume Born's condition that the function must be periodic in the z coordinate, that is $\psi_n(x,y,z+L_z) = \psi_n(x,y,z)$ one can write

$$E_{n,n'} = \hbar\omega_c\left(n + \frac{1}{2}\right) + \frac{1}{2m}\left(\frac{2\pi\hbar}{L_z}n'\right)^2, \quad n,n' \in \mathbb{Z}^+. \tag{2.51}$$

2.1.1 About the degeneration of the solution with Landau's gauge

Until now, it seems like we are done with the whole situation and the only thing left is to build up the general solution by doing the superposition of all the posible states, however, there is still one thing left and is the study of the system's degeneration. To begin with it, is useful to start with the expression (2.21) but, for simplicity, limited to the two dimensional case, that is, on the plane (x, y)

$$\hat{\mathbf{H}} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 + 2m\omega_c y\hat{p}_x + m^2\omega_c^2 y^2\right), \tag{2.52}$$

then we recall the following commutation identities, first lets denote the positions as $x = x_1$, $y = x_2$ and $z = z_3$ and the momentum operators as $\hat{p}_x = \hat{p}_1$, $\hat{p}_y = \hat{p}_2$ and $\hat{p}_z = \hat{p}_3$, then we can write

$$[\hat{p}_i, \hat{p}_j] = 0, \quad i, j = 1, 2, 3, \tag{2.53}$$

$$[x_l, \hat{p}_j] = i\hbar\delta_{l,j}, \quad l, j = 1, 2, 3, \tag{2.54}$$

where $\delta_{l,j}$ is Kronecker's delta. Then, being $\hat{A}$, $\hat{B}$ and $\hat{C}$ three operators, it follows that

$$[AB, C] = A[B, C] + [A, C]B, \tag{2.55}$$

using this identities is easy to prove that

$$[\hat{p}_x, \hat{\mathbf{H}}] = 0. \tag{2.56}$$

This implies that the operator $\hat{p}_x$ is conserved. Normally one can believe that if an operator is conserved, then, necessarily it shares basis with the Hamiltonian, or in other words, the action of the conserved operator on the eigenfunctions of the Hamiltonian is such that the resulting function is proportional to itself, that is, being ψ an eigenfunction of the Hamiltonian, $\hat{H}\psi = E\psi$ then, the action of the conserved operator on the function is such that $\hat{p}_x\psi \sim \psi$. However, this assumption is not generally correct and the reason of it is because one can not simply assume how the action of the operators will be. To clarify this situation lets rewrite expression (2.56) acting on an eigenfunction ψ

$$\hat{\mathbf{H}}\left(\hat{p}_x\psi\right) = \hat{p}_x\left(\hat{\mathbf{H}}\psi\right), \tag{2.57}$$

then, we can write

$$\hat{\mathbf{H}}\left(\hat{p}_x\psi\right) = E\hat{p}_x\psi, \tag{2.58}$$

of course the above expression is satisfied if one assume that $\hat{p}_x\psi = A\psi$ where A is a constant, however, there is a second option that can satisfy the above equality too, that is that the action of the conserved operator on the function gives a different function (denoted by f) who is still an eigenfunction of the Hamiltonian. That is, if

$$\hat{p}_x\psi = f, \tag{2.59}$$

then

$$\hat{\mathbf{H}}f = Ef. \tag{2.60}$$

This second observation is, in fact, our current situation. In particular, the action of $\hat{p}_x$ over the solution (2.49) when $z = 0$ is as follows

$$\hat{p}_x\psi_n(x,y) = -m\omega_c y\psi_n - \frac{i\hbar}{\sqrt{L_y L_z}}\exp\left(-i\frac{m\omega_c}{\hbar}xy\right)\frac{\partial}{\partial x}\phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}x\right), \tag{2.61}$$

now, we use the following expression to calculate the harmonic oscillator derivative

$$\frac{\partial}{\partial\xi}\phi_n(\xi) = -\xi\phi_n + \sqrt{2n}\phi_{n-1}, \tag{2.62}$$

using the change of variable, $\xi = \sqrt{\frac{m\omega_c}{\hbar}}x$, we can write

$$\hat{p}_x\psi_n(x,y) = m\omega_c(ix - y)\psi_n - i\sqrt{2nm\omega_c\hbar}\psi_{n-1} \tag{2.63}$$

thus, as we previously said, by applying the conserved operator we have a new function $f_n(x,y) = \hat{p}_x\psi_n(x,y)$ which is also an eigenfunction. This last statement can be proved by substituting this new function in the Hamiltonian (2.52)

$$\hat{\mathbf{H}} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 + 2m\omega_c y\hat{p}_x + m^2\omega_c^2 y^2\right). \tag{2.64}$$

To complete this task is useful to write down some expressions

$$\hat{p}_x(x\psi_n) = -i\hbar\psi_n + x\hat{p}_x\psi_n, \tag{2.65}$$

$$\hat{p}_x^2(x\psi_n) = -2i\hbar\hat{p}_x\psi_n + x\hat{p}_x^2\psi_n \tag{2.66}$$

and

$$\hat{p}_y^2(y\psi_n) = -2i\hbar\hat{p}_y\psi_n + y\hat{p}_y^2\psi_n, \tag{2.67}$$

then applying the Hamiltonian to the new function we can write

$$\hat{\mathbf{H}}f_n = m\omega_c\Big(i\hat{\mathbf{H}}(x\psi_n) - \hat{\mathbf{H}}(y\psi_n)\Big) - i\sqrt{2nm\omega_c\hbar}\hat{\mathbf{H}}\psi_{n-1} \tag{2.68}$$

using the equalities (2.65), (2.66) and (2.67) is possible to write down the following results

$$\hat{\mathbf{H}}(x\psi_n) = x\hat{\mathbf{H}}\psi_n + \hbar\omega_c x\psi_n - \frac{\hbar}{m}\sqrt{2nm\omega_c\hbar}\psi_{n-1}, \tag{2.69}$$

and

$$\hat{\mathbf{H}}(y\psi_n) = y\hat{\mathbf{H}}\psi_n + i\hbar\omega_c x\psi_n, \tag{2.70}$$

substituting them in expression (2.68)

$$\hat{\mathbf{H}}f_n = m\omega_c\left(ix\hat{\mathbf{H}}\psi_n + i\hbar\omega_c x\psi_n - i\frac{\hbar}{m}\sqrt{2nm\omega_c\hbar}\psi_{n-1} - y\hat{\mathbf{H}}\psi_n - i\hbar\omega_c x\psi_n\right) - i\sqrt{2nm\omega_c\hbar}\hat{\mathbf{H}}\psi_{n-1} \tag{2.71}$$

using the fact that ψ_n has eigenvalues of

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right), \tag{2.72}$$

and rearranging, we write down the expression

$$\hat{\mathbf{H}}f_n = E_n m\omega_c(ix - y)\psi_n - i\sqrt{2nm\omega_c\hbar}(E_{n-1} + \hbar\omega_c)\psi_{n-1}, \tag{2.73}$$

then realizing that

$$E_{n-1} + \hbar\omega_c = E_n, \tag{2.74}$$

we have

$$\hat{\mathbf{H}}f_n = E_n\left(m\omega_c(ix - y)\psi_n - i\sqrt{2nm\omega_c\hbar}\psi_{n-1}\right), \tag{2.75}$$

hence, we have proved that $f_n(x, y)$ is another eigenfunction, that is

$$\hat{\mathbf{H}}f_n = E_n f_n. \tag{2.76}$$

Finally, one can make the observation that the expression (2.56) can be generalized to any number of applications of the operator $\hat{p}_x$. So, being $j \in \mathbb{Z}^+$ we denote the $j-$application of an operator as

$$\hat{p}_x^j = \hat{p}_x \circ \hat{p}_x \circ ...\hat{p}_x, \quad j\text{-times}, \tag{2.77}$$

using the identity (2.55) is easy to prove that

$$[\hat{p}_x^j, \hat{\mathbf{H}}] = 0, \tag{2.78}$$

consequently we have

$$\hat{\mathbf{H}}\left(\hat{p}_x^j\psi\right) = E\hat{p}_x^j\psi. \tag{2.79}$$

This means that we can have a numerable set of j eigenfunctions, as many as times we apply the conserved operator. First, we are going to denote the new generated function as

$$f_n^j(x,y) = \hat{p}_x^j\psi_n, \qquad j = 0,1,2,... \tag{2.80}$$

where the definition $\hat{p}_x^0 = 1$ was made. Then, applying j-times more the operator $\hat{p}_x$ to the expression (2.63) we have

$$f_n^{j+1} = m\omega_c\hat{p}_x^j\left((ix-y)\psi_n\right) - i\sqrt{2nm\omega_c\hbar}\hat{p}_x^j\psi_{n-1} \tag{2.81}$$

using the formula for the generalization of the derivative of the product of two functions

$$\frac{d^j}{dx^j}\left(f(x)g(x)\right) = \sum_{m=0}^{j}\binom{j}{m}\frac{d^{j-m}}{dx^{j-m}}(f(x))\frac{d^m}{dx^m}(g(x)) \tag{2.82}$$

where

$$\binom{j}{m} = \frac{j!}{m!(j-m)!}, \tag{2.83}$$

is the binomial coefficient. A careful analysis show us that the only not zero terms of the sum are the ones when $m = j, j-1$, we can write

$$\hat{p}_x^j\left((ix-y)\psi_n\right) = (ix-y)\hat{p}_x^j\psi_n + \hbar j\hat{p}_x^{j-1}\psi_n, \tag{2.84}$$

finally, we got the following expression for the eigenfunctions

$$f_n^{j+1}(x,y) = m\omega_c\left(\hbar j f_n^{j-1} + (ix-y)f_n^j\right) - i\sqrt{2nm\omega_c\hbar}f_{n-1}^j. \tag{2.85}$$

To end with this section we must say that the general solution of the two dimensional system can be written as

$$\Psi(x,y) = \sum_{n,j} C_{n,j}f_n^j. \tag{2.86}$$

2.1.2 Operators associated to the index of degeneration j for Landau's gauge

In the appendix (A.2) is shown that the functions (2.85) $f_n^{j+1}(x,y)$ satisfy the eigenvalue equation

$$\hat{\mathbf{H}} f_n^{j+1} = E_n f_n^{j+1}, \qquad E_n = \hbar\omega_c \left(n + \frac{1}{2}\right), \tag{2.87}$$

if the expression

$$m\omega_c \hbar j f_n^{j-1} + m\omega_c i x f_n^j + i\hat{p}_y f_n^j = 0, \tag{2.88}$$

is satisfied. A quick corroboration of the above equation can be made by applying the operator $\hat{p}_y$ to the equation (2.49)

$$\hat{p}_y \psi_n = -m\omega_c x \psi_n, \tag{2.89}$$

then, we apply the operator $\hat{p}_x$ j-th times, use the fact that $[\hat{p}_y, \hat{p}_x^j] = 0$ and the definition (2.80) to write down the following expression

$$\hat{p}_y f_n^j = -m\omega_c \hat{p}_x^j (x\psi_n), \tag{2.90}$$

finally, we use the formula (2.82) and realize that the only no zero terms are the ones when $m = j, j-1$ and we obtain

$$\hat{p}_y f_n^j = -m\omega_c (x f_n^j - i\hbar j f_n^{j-1}), \tag{2.91}$$

which is the expression (2.88). We can rewrite the above expression in a very curios way to reveal a new eigenvalue equality

$$(\hat{p}_y + m\omega_c x)\hat{p}_x f_n^{j-1} = i m\omega_c \hbar j f_n^{j-1}, \tag{2.92}$$

or making $j = j' + 1$ (and renaming j' as j), it follows that

$$(\hat{p}_y + m\omega_c x)\hat{p}_x f_n^j = i m\omega_c \hbar (j+1) f_n^j. \tag{2.93}$$

The above equality define us a new operator that has imaginary eigenvalues, therefore the operator $(\hat{p}_y + m\omega_c x)\hat{p}_x$ is not Hermitian. Is not difficult to prove that this new operator is, in fact, conserved, however if one wants to know about the physical meaning of this operator, is necessary to inquiry a little bit more.
Using our initial gauge (2.18)

$$\mathbf{A} = B(-y, 0, 0), \tag{2.94}$$

which define a constant magnetic field in the positive z direction

$$\mathbf{B} = \nabla \times \mathbf{A} = B\hat{k}, \tag{2.95}$$

then, using the results (2.14), (2.15) and (2.16) we can define the momentum operators as

$$\hat{\pi}_x = \hat{p}_x + m\omega_c y, \quad \hat{\pi}_y = \hat{p}_y, \quad \hat{\pi}_z = \hat{p}_z \tag{2.96}$$

which can be used to build up the Hamiltonian operator as follows

$$\hat{\mathbf{H}} = \frac{1}{2m}\left(\hat{\pi}_x^2 + \hat{\pi}_y^2 + \hat{\pi}_z^2\right). \tag{2.97}$$

On the other hand, we can select the alternative *Landau's* gauge

$$\mathbf{A}' = B(0, -x, 0) \tag{2.98}$$

which define a constant magnetic field of the same magnitud but in opposite direction, that is

$$\mathbf{B}' = \nabla \times \mathbf{A}' = -B\hat{k}, \tag{2.99}$$

similarly, using the results (2.14), (2.15) and (2.16), this will define us a set of momentum operators

$$\hat{\pi}'_x = \hat{p}_x, \quad \hat{\pi}'_y = \hat{p}_y + m\omega_c x, \quad \hat{\pi}'_z = \hat{p}_z, \tag{2.100}$$

then, using the commutation properties (2.53) and (2.54), is easy to prove that the next set of commutation relations holds

$$[\hat{\pi}'_x, \hat{\pi}_x] = [\hat{\pi}'_x, \hat{\pi}_y] = [\hat{\pi}'_x, \hat{\pi}_z] = 0, \tag{2.101}$$

$$[\hat{\pi}'_y, \hat{\pi}_x] = [\hat{\pi}'_y, \hat{\pi}_y] = [\hat{\pi}'_y, \hat{\pi}_z] = 0, \tag{2.102}$$

and

$$[\hat{\pi}'_z, \hat{\pi}_x] = [\hat{\pi}'_z, \hat{\pi}_y] = [\hat{\pi}'_z, \hat{\pi}_z] = 0. \tag{2.103}$$

With the aid of the above relations and (2.55) is straightforward to prove that

$$[\hat{\pi}'_x, \hat{\mathbf{H}}] = [\hat{\pi}'_y, \hat{\mathbf{H}}] = [\hat{\pi}'_z, \hat{\mathbf{H}}] = 0. \tag{2.104}$$

Hence, the eigenvalue relation (2.93) can be written as

$$\hat{\pi}'_y \hat{\pi}'_x f_n^j = im\omega_c\hbar(j+1) f_n^j, \tag{2.105}$$

and using the results (2.104) is easy to prove that the operator $\hat{O} = \hat{\pi}'_y \hat{\pi}'_x$ is conserved,

$$[\hat{O}, \hat{\mathbf{H}}] = [\hat{\pi}'_y \hat{\pi}'_x, \hat{\mathbf{H}}] = \hat{\pi}'_y [\hat{\pi}'_x, \hat{\mathbf{H}}] + [\hat{\pi}'_y, \hat{\mathbf{H}}] \hat{\pi}'_x = 0. \tag{2.106}$$

Therefore, we can see that there is a relation between the degeneration of the system and the inversion of the direction of the magnetic field. Finally, one must point out that due to the discussion of the prior subsection, one could think that if the operator $\hat{\pi}'_y$ is also conserved then it must be another eigenfunction generator, like in the case of $\hat{\pi}'_x$, however, is not difficult to prove that

$$\hat{\pi}'_y \psi_n(x, y) = 0, \tag{2.107}$$

hence, we see that it does not generate any new function.

2.1.3 Summary of the results with Landau's gauge

To end with the analysis of the Hamiltonian with Landau's gauge, we present the final results in a brief way. Being Landau's gauge

$$\mathbf{A} = B(-y, 0, 0), \tag{2.108}$$

the Hamiltonian is

$$\hat{\mathbf{H}} = \frac{1}{2m} \left((\hat{p}_x + m\omega_c y)^2 + \hat{p}_y^2 + \hat{p}_z^2 \right), \quad \text{where} \quad \omega_c = \frac{qB}{mc}, \tag{2.109}$$

the solutions of Schrödinger's eigenvalue equation are written as

$$f_{n,n'}^{j+1}(x, y, z) = f_n^{j+1}(x, y) \exp\left(i \frac{2\pi}{L_z} n' z \right), \tag{2.110}$$

where

$$f_n^{j+1}(x, y) = m\omega_c \left(\hbar j f_n^{j-1} + (ix - y) f_n^j \right) - i\sqrt{2nm\omega_c\hbar} f_{n-1}^j, \tag{2.111}$$

where the following definitions were made

$$f_n^0(x, y) = \frac{1}{\sqrt{L_y L_z}} \exp\left(-i \frac{m\omega_c}{\hbar} xy \right) \phi_n \left(\sqrt{\frac{m\omega_c}{\hbar}} x \right), \tag{2.112}$$

$$f_n^j(x, y, z) = \hat{p}_x^j f_n^0, \qquad j = 0, 1, 2, ... \tag{2.113}$$

and

$$\phi_n(\chi) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega_c}{\pi\hbar} \right)^{1/4} \exp\left(-\frac{\chi^2}{2} \right) H_n(\chi), \tag{2.114}$$

is the harmonic oscillator solution. The prior expressions satisfies the couple of eigenvalue

$$\hat{\mathbf{H}} f^j_{n,n'} = E_{n,n'} f^j_{n,n'}, \tag{2.115}$$

where

$$E_{n,n'} = \hbar\omega_c \left(n + \frac{1}{2}\right) + \frac{1}{2m}\left(\frac{2\pi\hbar}{L_z} n'\right)^2, \quad n, n' \in \mathbb{Z}^+. \tag{2.116}$$

and

$$\hat{\pi}'_y \hat{\pi}'_x f^j_{n,n'} = i m\omega_c \hbar (j+1) f^j_{n,n'}, \tag{2.117}$$

where

$$\hat{\pi}'_x = \hat{p}_x, \quad \hat{\pi}'_y = \hat{p}_y + m\omega_c x. \tag{2.118}$$

Therefore, the general solution of Schrödinger's equation is

$$\Psi(x, y, z, t) = \sum_{n,n',j} C_{n,n',j} f^j_n(x, y) \exp\left(i\frac{2\pi}{L_z} n' z - i\frac{E_{n,n'}}{\hbar} t\right), \tag{2.119}$$

where $C_{n,n',j}$ are complex constants.

2.2 Hamiltonian with Symmetric gauge

It is known that the selection of the gauge is not unique, one can choose $\mathbf{A}' = \mathbf{A} + \nabla S$, where S is a scalar function. Note that if we choose

$$\mathbf{A} = \frac{B}{2}(-y, x, 0), \tag{2.120}$$

the description of the magnetic field is the same than in the prior case where we chose of *Landau's* gauge, that is, it describes a constant magnetic field of magnitud B along the positive z direction, or mathematically,

$$\mathbf{B} = \nabla \times \mathbf{A} = B\hat{k}. \tag{2.121}$$

However, even though if it describes the same classical system, it turns out that the quantum systems are quite different. The selection of the gauge as (2.120) is known as *symmetric gauge*. Using this vector potential the Hamiltonian is written as

$$\hat{\mathbf{H}} = \frac{1}{2m}\left(\left(\hat{p}_x + \frac{m\omega_c}{2}y\right)^2 + \left(\hat{p}_y - \frac{m\omega_c}{2}x\right)^2 + \hat{p}_z^2\right). \tag{2.122}$$

Similar to the prior case, is not difficult to corroborate that Schrödinger's equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{\mathbf{H}}\psi, \tag{2.123}$$

is separable in both variables time and z coordinate, therefore, we can write the solution as

$$\psi(x, y, z, t) = \varphi(x, y)\exp\left(i\frac{\sqrt{2mE_2}}{\hbar}z - i\frac{E}{\hbar}t\right), \tag{2.124}$$

where

$$\hat{\mathbf{H}}\psi = E\psi, \quad E = E_1 + E_2, \tag{2.125}$$

and, likewise the prior case, we can use Born periodicity condition in z direction and determine the energy E_2 as

$$E_2 = \frac{1}{2m}\left(\frac{2\pi\hbar}{L_z}n'\right)^2, \quad n' \in \mathbb{Z}^+. \tag{2.126}$$

By doing this we can focus in the search of the solution on the (x, y) plane which is describe by the two dimensional Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2m}\left(\left(\hat{p}_x + \frac{m\omega_c}{2}y\right)^2 + \left(\hat{p}_y - \frac{m\omega_c}{2}x\right)^2\right), \tag{2.127}$$

such that

$$\hat{\mathcal{H}}\varphi = E_1\varphi. \tag{2.128}$$

By rewriting the squared binomials terms and using (2.54) the Hamiltonian takes the form of

$$\hat{\mathcal{H}} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 - m\omega_c\hat{L}_z + \frac{m^2\omega_c^2}{4}(x^2+y^2)\right), \tag{2.129}$$

where we used the definition of the angular momentum operator in z direction, that is

$$\hat{L}_z = x\hat{p}_y - y\hat{p}_x. \tag{2.130}$$

Hence, substituting the differential form of the operators the eigenvalue partial differential equation can be written as

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} - i\frac{m\omega_c}{\hbar}\left(x\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x}\right) - \frac{m^2\omega_c^2}{4\hbar^2}(x^2+y^2)\varphi\right) = E_1\varphi, \tag{2.131}$$

note that this expression can not be separated in cartesian, (x, y), nor polar coordinates, (r, θ), therefore, is necessary to choose a couple of independent variable that could help to solve the above equation. The problem is simplified if we work in the complex plane, that is, we make the change of variable

$$z = x + iy, \quad z^* = x - iy, \tag{2.132}$$

with this selection of variables the differential operators are changed as

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}, \tag{2.133}$$

and

$$\frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right), \tag{2.134}$$

then, the following expressions can be calculated

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\partial z^*}, \tag{2.135}$$

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = i\left(z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*}\right), \tag{2.136}$$

then, defining the constant

$$\alpha = \frac{m\omega_c}{4\hbar}, \tag{2.137}$$

the eq.(2.131) takes the form

$$-\frac{2\hbar^2}{m}\left(\frac{\partial^2\varphi}{\partial z\partial z^*} + \alpha\left(z\frac{\partial\varphi}{\partial z} - z^*\frac{\partial\varphi}{\partial z^*}\right) - \alpha^2 zz^*\varphi\right) = E_1\varphi. \tag{2.138}$$

Following up on this, we can simplify even more the problem by proposing the solution of the form

$$\varphi(z, z^*) = e^{-\alpha zz^*}\phi(z, z^*), \tag{2.139}$$

substituting this in the above partial differential equation, simplifying and rearranging, one can write

$$\frac{\partial^2\phi}{\partial z\partial z^*} - 2\alpha z^*\frac{\partial\phi}{\partial z^*} = \left(\alpha - \frac{mE_1}{2\hbar^2}\right)\phi. \tag{2.140}$$

This last expression is separable if we propose the function ϕ to be written as

$$\phi(z, z^*) = f(z)g(z^*), \tag{2.141}$$

substituting in the last expression we have that

$$\frac{\partial f}{\partial z}\frac{\partial g}{\partial z^*} - 2\alpha z^* f\frac{\partial g}{\partial z^*} = \left(\alpha - \frac{mE_1}{2\hbar^2}\right)fg, \tag{2.142}$$

dividing by $f(\partial g/\partial z^*)$

$$\frac{1}{f}\frac{\partial f}{\partial z} = 2\alpha z^* + \left(\alpha - \frac{mE_1}{2\hbar^2}\right)\frac{g}{\frac{\partial g}{\partial z^*}}. \tag{2.143}$$

Continuing with the analysis, note that the left hand side of the above expression as well as the right hand side are complex expressions with domain and image in the complex numbers, that is, both expressions takes elements in the complex plane $\mathbb{C}$ and maps them to the complex plane again $\mathbb{C}$,

$$\frac{1}{f}\frac{\partial f}{\partial z} : \mathbb{C} \to \mathbb{C}, \tag{2.144}$$

$$2\alpha z^* + \left(\alpha - \frac{mE_1}{2\hbar^2}\right)\frac{g}{\frac{\partial g}{\partial z^*}} : \mathbb{C} \to \mathbb{C}, \tag{2.145}$$

therefore, the expression (2.143) is satisfied if is equal to a complex constant called λ, then we have the following couple of differential equations

$$\frac{\partial f}{\partial z} = -\lambda f, \tag{2.146}$$

and

$$2\alpha z^* \frac{\partial g}{\partial z^*} + \left(\alpha - \frac{mE_1}{2\hbar^2}\right) g = -\lambda \frac{\partial g}{\partial z^*}. \tag{2.147}$$

Note that we have used a minus sign for the constant λ which has no other purpose than the mere esthetic in the final form of the solution. The solutions of the above couple of equations is obtain in a straightforward integration and the reader can corroborate that the following solutions are obtained

$$f(z) = e^{-\lambda z}, \tag{2.148}$$

and

$$g(z^*) = \left(2\alpha z^* + \lambda\right)^{\frac{1}{2\alpha}\left(\frac{mE_1}{2\hbar^2} - \alpha\right)}. \tag{2.149}$$

Now that we have the eigenfunction, we need to determine the energy E_1. To complete this task, we continue analyzing the expression (2.147), we will now study it as a power series of the variable z^*, that is

$$g(z^*) = \sum_{n=0}^{\infty} a_n (z^*)^n, \tag{2.150}$$

where $a_n \in \mathbb{C}$. Then the first derivative can be written as

$$\frac{\partial g}{\partial z^*} = \sum_{n=0}^{\infty} n a_n (z^*)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (z^*)^n, \tag{2.151}$$

substituting (2.150), (2.151) in (2.147) and rearranging the result we find out that

$$\sum_{n=0}^{\infty} \left(\lambda(n+1)a_{n+1} + \left(2\alpha n + \alpha - \frac{mE_1}{2\hbar^2}\right) a_n\right) (z^*)^n = 0 \tag{2.152}$$

therefore, this expression give us the the following recurrence relation for the coefficients a_n

$$a_{n+1} = \frac{\frac{mE_1}{2\hbar^2} - 2\alpha n - \alpha}{\lambda(n+1)} a_n. \tag{2.153}$$

From this last expression we can see that the asymptotic behavior is such that for $n >> 1$ we have

$$\frac{a_{n+1}}{a_n} \to -\frac{2\alpha}{\lambda} \tag{2.154}$$

this means that at some $n = n_0 >> 1$ the coefficients can be written as $a_{n_0+1} \sim -2\alpha a_{n_0}/\lambda$ and the series (2.150) takes the form of

$$g(z^*) = \sum_{n=0}^{n_0} a_n (z^*)^n + a_{n_0} \sum_{n=n_0+1}^{\infty} \left(-\frac{2\alpha}{\lambda}\right)^n (z^*)^n, \tag{2.155}$$

this serie diverges when $|2\alpha z^*|/|\lambda| > 1$. Therefore, in order to have polynomial solutions, we need to cut off the series (2.153), this can be done by doing

$$\frac{mE_1}{2\hbar^2} - 2\alpha n - \alpha = 0, \tag{2.156}$$

substituting eq.(2.137) and solving for E_1 we have that

$$E_1 = \hbar\omega_c \left(n + \frac{1}{2}\right). \tag{2.157}$$

Finally, using this result in eq. (2.149) we have that

$$g(z^*) = (2\alpha z^* + \lambda)^n, \tag{2.158}$$

and the solution in cartesian coordinates is

$$\varphi_n(x, y) = A_n \exp\left(-\alpha(x^2 + y^2) - \lambda(x + iy)\right) \left(2\alpha(x - iy) + \lambda\right)^n, \tag{2.159}$$

where A_n is the normalization constant and is defined as

$$A_n = \frac{e^{-|\lambda|^2/4\alpha}}{\sqrt{(2\alpha)^{n-1}\pi n!}}, \tag{2.160}$$

the details of the calculation of this constant can be seen in appendix (B.1).

2.2.1 About the degeneration of symmetric gauge and its eigenfunctions generators

Similarly as we did in section (2.1.1), we look for conserved operators such that they can act as eigenfunctions generators. From the expression (2.129)

$$\hat{\mathcal{H}} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 - m\omega_c \hat{L}_z + \frac{m^2\omega_c^2}{4}(x^2 + y^2)\right), \tag{2.161}$$

is easy to prove that the angular momentum $\hat{L}_z$ is conserved, that is

$$[\hat{L}_z, \hat{\mathcal{H}}] = 0. \tag{2.162}$$

However, an interesting situation comes up for this system. From the discussion in section (2.1.2) we could ask, what would happen with the momentum operators when the magnetic field direction is inverted?, that is, for the gauge used here (2.120)

$$\mathbf{A} = \frac{B}{2}(-y, x, 0), \tag{2.163}$$

the momentum operators defined by equations (2.14) and (2.15) takes the form of

$$\hat{\pi}_x = \hat{p}_x + \frac{m\omega_c}{2}y, \qquad \hat{\pi}_y = \hat{p}_y - \frac{m\omega_c}{2}x. \tag{2.164}$$

Then, the gauge that describes a magnetic field of the same magnitud but oposite direction, that is,

$$\mathbf{B} = \nabla \times \mathbf{A}' = -B\hat{k}, \tag{2.165}$$

is given by

$$\mathbf{A}' = \frac{B}{2}(y, -x, 0), \tag{2.166}$$

en define us, via the equalities (2.14) and (2.15), the following momentum operators

$$\hat{\pi}'_x = \hat{p}_x - \frac{m\omega_c}{2}y, \qquad \hat{\pi}'_y = \hat{p}_y + \frac{m\omega_c}{2}x, \tag{2.167}$$

and the following commutation relations can be calculated using the properties (2.53) and (2.54)

$$[\hat{\pi}'_x, \hat{\pi}_x] = [\hat{\pi}'_x, \hat{\pi}_y] = 0, \tag{2.168}$$

and

$$[\hat{\pi}'_y, \hat{\pi}_x] = [\hat{\pi}'_y, \hat{\pi}_y] = 0. \tag{2.169}$$

Note that the Hamiltonian (2.129) can be written using the momentum operators as follows

$$\hat{\mathcal{H}} = \frac{1}{2m}\left(\hat{\pi}_x^2 + \hat{\pi}_y^2\right), \tag{2.170}$$

using this last expression along with the property (2.55) is straightforward calculation to deduce that

$$[\hat{\pi}'_x, \hat{\mathcal{H}}] = [\hat{\pi}'_y, \hat{\mathcal{H}}] = 0, \tag{2.171}$$

in other words, similar to the prior case when *Landau's* gauge was used, the inversion of the magnetic fields direction define the conserved momentum operators. Therefore, we have three conserved operators that are, actually, eigenfunction generators. The operator $\hat{\pi}'_x$ applied j-th times to the expression (2.159), that is,

$$(\hat{\pi}'_x)^j \varphi_n(x,y) = f_n^j(x,y), \quad j = 0,1,2... \tag{2.172}$$

where the definition $\varphi_n(x,y) = f_n^0(x,y)$ was made, gives the expression

$$f_n^{j+1} = \frac{m\omega_c}{2}\left((ix-y)f_n^j + j\hbar f_n^{j-1}\right) + i\hbar\lambda f_n^j - i\sqrt{\frac{m\omega_c}{2}}\hbar n f_{n-1}^j. \tag{2.173}$$

The operator $\hat{\pi}'_y$ applied j-th times to the expression (2.159), that is,

$$(\hat{\pi}'_y)^j \varphi_n(x,y) = g_n^j(x,y), \quad j = 0,1,2... \tag{2.174}$$

where the definition $\varphi_n(x,y) = g_n^0(x,y)$ was made, gives the expression

$$g_n^{j+1} = \frac{m\omega_c}{2}\left((x+iy)g_n^j + j\hbar g_n^{j-1}\right) - \hbar\lambda g_n^j - \sqrt{\frac{m\omega_c}{2}}\hbar n g_{n-1}^j. \tag{2.175}$$

Finally, the operator $\hat{L}_z$ applied j-th times to the expression (2.159), that is,

$$(\hat{L}_z)^j \varphi_n(x,y) = \mathcal{L}_n^j(x,y), \quad j = 0,1,2... \tag{2.176}$$

where the definition $\varphi_n(x,y) = \mathcal{L}_n^0(x,y)$ was made, gives the expression

$$\mathcal{L}_n^{j+1} = \sum_{m=0}^{j}\left(-\hbar\lambda(x+iy)c_m^j\mathcal{L}_n^m - \sqrt{\frac{m\omega_c\hbar n}{2}}(x-iy)d_m^j\mathcal{L}_{n-1}^m\right). \tag{2.177}$$

The details of the calculations of the eigenfunctions are shown in the appendix (B.2). All the above expressions satisfies the eigenvalue equation given by the Hamiltonian (2.129) with eigenvalues (2.157)

$$\hat{\mathcal{H}} f_n^j = E_n f_n^j, \tag{2.178}$$

$$\hat{\mathcal{H}} g_n^j = E_n g_n^j, \tag{2.179}$$

and

$$\hat{\mathcal{H}} \mathcal{L}_n^j = E_n \mathcal{L}_n^j. \tag{2.180}$$

Besides, similarly as we did in the case where we used the Landau's gauge, it is possible to find a second eigenvalue expression regarding the index of degeneration j. In the appendix (B.4) is shown that the commutation relation holds (B.93)

$$[\hat{\pi}'_y, (\hat{\pi}'_x)^{j+1}] = i\hbar m\omega_c(j+1)(\hat{\pi}'_x)^j. \tag{2.181}$$

applying it to the eigenfunction (2.159) and using the definition (2.172)

$$\hat{\pi}'_y(\hat{\pi}'_x)f_n^j - (\hat{\pi}'_x)^{j+1}\hat{\pi}'_y f_n^0 = i\hbar m\omega_c(j+1)f_n^j, \tag{2.182}$$

then, from the eigenfunctions (2.173) and (2.175) for $j=0$, is not difficult to figure out that

$$\hat{\pi}'_y f_n^0 = -2\hbar\lambda f_n^0 - i(\hat{\pi}'_x)f_n^0. \tag{2.183}$$

hence, substituting this last expression in (2.182) we can write the second eigenvalue equation as

$$(\hat{\pi}'_y + i\hat{\pi}'_x + 2\hbar\lambda)\hat{\pi}'_x f_n^j = i\hbar m\omega_c(j+1)f_n^j. \tag{2.184}$$

This last expression define a new operator

$$\hat{O}_1 = (\hat{\pi}'_y + i\hat{\pi}'_x + 2\hbar\lambda)\hat{\pi}'_x, \tag{2.185}$$

and using the commutation identities (2.168), (2.169) and (2.55) is straightforward to prove that this operator is conserved, that is

$$[\hat{O}_1, \hat{\mathcal{H}}] = 0. \tag{2.186}$$

Alternatively, the commutation (B.96) can be used to get a second eigenvalue expression for the functions (2.175)

$$[(\hat{\pi}'_y)^{j+1}, \hat{\pi}'_x] = i\hbar m\omega_c(j+1)(\hat{\pi}'_y)^j, \tag{2.187}$$

using (2.183) knowing that $f_n^0 = g_n^0$ one can realize that

$$(i\hat{\pi}'_y - \hat{\pi}'_x + 2i\hbar\lambda)\hat{\pi}'_y g_n^j = i\hbar m\omega_c(j+1)g_n^j, \tag{2.188}$$

which define a new operator

$$\hat{O}_2 = (i\hat{\pi}'_y - \hat{\pi}'_x + 2i\hbar\lambda)\hat{\pi}'_y, \tag{2.189}$$

that is also conserved, that is,

$$[\hat{O}_2, \hat{\mathcal{H}}] = 0. \tag{2.190}$$

We must point out at some observations, note that the eigenvalues of the operators $\hat{O}_1$ and $\hat{O}_2$ are the same eigenvalues given by (2.105), its eigenvalues are imaginaries, therefore, there are not Hermitian operators and is posible to express the angular momentum in terms of the operators (2.167) as follows

$$\hat{L}_z = x\hat{\pi}'_y - y\hat{\pi}'_x - \frac{m\omega_c}{2}(x^2+y^2). \tag{2.191}$$

2.2.2 Summary of the results with the symmetric gauge

To conclude with this analysis, we present the final results in a brief way. Being the symmetric gauge (2.120)

$$\mathbf{A} = \frac{B}{2}(-y, x, 0), \tag{2.192}$$

this define the Hamiltonian

$$\hat{\mathbf{H}} = \frac{1}{2m}\left(\hat{\mathcal{H}} + \hat{p}_z^2\right), \tag{2.193}$$

where

$$\hat{\mathcal{H}} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 - m\omega_c\hat{L}_z + \frac{m^2\omega_c^2}{4}(x^2+y^2)\right). \tag{2.194}$$

Making the definition $\varphi_n(x,y) = f_n^0(x,y) = g_n^0(x,y) = \mathcal{L}_n^0(x,y)$ where,

$$\varphi_n(x,y) = \frac{e^{-|\lambda|^2/4\alpha}}{\sqrt{(2\alpha)^{n-1}\pi n!}}\exp\left(-\alpha(x^2+y^2) - \lambda(x+iy)\right)\left(2\alpha(x-iy)+\lambda\right)^n, \tag{2.195}$$

the degeneration for the two dimensional Hamiltonian, $\hat{\mathcal{H}}$, are given by the application of the three conserved operators

$$\hat{\pi}'_x = \hat{p}_x - \frac{m\omega_c}{2}y, \qquad \hat{\pi}'_y = \hat{p}_y + \frac{m\omega_c}{2}x, \qquad \hat{L}_z = x\hat{p}_y - y\hat{p}_x \tag{2.196}$$

which define us the set of eigenfunctions

$$(\hat{\pi}'_x)^j\varphi_n(x,y) = f_n^j(x,y), \quad j = 0,1,2... \tag{2.197}$$

$$(\hat{\pi}'_y)^j\varphi_n(x,y) = g_n^j(x,y), \quad j = 0,1,2... \tag{2.198}$$

$$(\hat{L}_z)^j\varphi_n(x,y) = \mathcal{L}_n^j(x,y), \quad j = 0,1,2... \tag{2.199}$$

or, written explicitly

$$f_n^{j+1} = \frac{m\omega_c}{2}\left((ix-y)f_n^j + j\hbar f_n^{j-1}\right) + i\hbar\lambda f_n^j - i\sqrt{\frac{m\omega_c}{2}}\hbar n f_{n-1}^j, \tag{2.200}$$

$$g_n^{j+1} = \frac{m\omega_c}{2}\left((x+iy)g_n^j + j\hbar g_n^{j-1}\right) - \hbar\lambda g_n^j - \sqrt{\frac{m\omega_c}{2}}\hbar n g_{n-1}^j, \tag{2.201}$$

and

$$\mathcal{L}_n^{j+1} = \sum_{m=0}^{j}\left(-\hbar\lambda(x+iy)c_m^j\mathcal{L}_n^m - \sqrt{\frac{m\omega_c\hbar n}{2}}(x-iy)d_m^j\mathcal{L}_{n-1}^m\right), \tag{2.202}$$

where all the above expressions satisfies the eigenvalue equation

$$\hat{\mathcal{H}}f_n^{j+1} = (E_1)_n f_n^{j+1}, \qquad \hat{\mathcal{H}}g_n^{j+1} = (E_1)_n g_n^{j+1}, \qquad \hat{\mathcal{H}}\mathcal{L}_n^{j+1} = (E_1)_n\mathcal{L}_n^{j+1} \tag{2.203}$$

having the eigenvalues

$$(E_1)_n = \hbar\omega_c\left(n+\frac{1}{2}\right), \tag{2.204}$$

besides, a second eigenvalue equation regarding the index of degeneration, j, is satisfied

$$(\hat{\pi}'_y + i\hat{\pi}'_x + 2\hbar\lambda)\hat{\pi}'_x f_n^j = i\hbar m\omega_c(j+1)f_n^j \qquad (i\hat{\pi}'_y - \hat{\pi}'_x + 2i\hbar\lambda)\hat{\pi}'_y g_n^j = i\hbar m\omega_c(j+1)g_n^j. \tag{2.205}$$

Therefore, the solutions for the Hamiltonian (2.193), can be written as

$$f_{n,n'}^{j+1}(x,y,z) = f_n^{j+1}(x,y)\exp\left(i\frac{2\pi}{L_z}n'z\right), \tag{2.206}$$

$$g_{n,n'}^{j+1}(x,y,z) = g_n^{j+1}(x,y)\exp\left(i\frac{2\pi}{L_z}n'z\right), \tag{2.207}$$

and

$$\mathcal{L}_{n,n'}^{j+1}(x,y,z) = \mathcal{L}_n^{j+1}(x,y)\exp\left(i\frac{2\pi}{L_z}n'z\right), \tag{2.208}$$

such that

$$\hat{\mathbf{H}}f_{n,n'}^{j+1} = E_{n,n'}f_{n,n'}^{j+1}, \qquad \hat{\mathbf{H}}g_{n,n'}^{j+1} = E_{n,n'}g_{n,n'}^{j+1}, \qquad \hat{\mathbf{H}}\mathcal{L}_{n,n'}^{j+1} = E_{n,n'}\mathcal{L}_{n,n'}^{j+1}. \tag{2.209}$$

having the eigenvalues

$$E_{n,n'} = \hbar\omega_c \left(n + \frac{1}{2}\right) + \frac{1}{2m}\left(\frac{2\pi\hbar}{L_z}n'\right)^2, \quad n, n' \in \mathbb{Z}^+. \tag{2.210}$$

Finally, the general solution for time dependent Schrödinger's equation can be written using any of the degenerated eigenfunctions

$$\Psi(x, y, z, t) = \sum_{n,n',j} A_{n,n',j} f_n^j(x, y) \exp\left(i\frac{2\pi}{L_z}n'z - i\frac{E_{n,n'}}{\hbar}t\right), \tag{2.211}$$

or

$$\Psi(x, y, z, t) = \sum_{n,n',j} B_{n,n',j} g_n^j(x, y) \exp\left(i\frac{2\pi}{L_z}n'z - i\frac{E_{n,n'}}{\hbar}t\right), \tag{2.212}$$

or

$$\Psi(x, y, z, t) = \sum_{n,n',j} C_{n,n',j} \mathcal{L}_n^j(x, y) \exp\left(i\frac{2\pi}{L_z}n'z - i\frac{E_{n,n'}}{\hbar}t\right), \tag{2.213}$$

where $A_{n,n',j}$, $B_{n,n',j}$ and $C_{n,n',j}$ are complex constants.

2.3 Charged particle with electromagnetic field

Until now, we have only analyzed the Schrödinger's equation with static magnetic field, however, in this section we are going to complicate the case a little bit by adding the influence of a constant electric field. We begin with the Hamiltonian (2.6)

$$\hat{\mathbf{H}} = \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{q}{c}\mathbf{A}\right)^2 + V, \tag{2.214}$$

where the magnetic field is given, as usual, by

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{2.215}$$

and the escalar function, $V = V(x, y, z)$, is such that $V = q\phi$ where $\phi = \phi(x, y, z)$ is the electric field potential such that the electric field is given as

$$\mathbf{E} = -\nabla\phi. \tag{2.216}$$

Then, we can select the potential such that the magnetic field is perpendicular to the electric field, $\mathbf{B} \perp \mathbf{E}$ which is, actually, the case we are interested in. If we use Landau's gauge as in section (2.1)

$$\mathbf{A} = B(-y, 0, 0), \tag{2.217}$$

then, a selection of the electric field potential is

$$\phi = -\mathcal{E}y, \tag{2.218}$$

where $\mathcal{E} \in \mathbb{R}^+$ is the electric field magnitud and is a constant, note that $\mathbf{B} \cdot \mathbf{E} = 0$ as desired. Then, the Hamiltontian to work with is written as

$$\hat{\mathbf{H}} = \frac{1}{2m}\left((\hat{p}_x + m\omega_c y)^2 + \hat{p}_y^2 + \hat{p}_z^2\right) - q\mathcal{E}y, \tag{2.219}$$

where cyclotron frequency is $\omega_c = qB/mc$, as in section (2.1). The above Hamiltonian can be rewritten as we have done while analyzing the case where we use the Landau's gauge only, eq.(2.21), that is

$$\hat{\mathbf{H}} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 + 2m\omega_c y\hat{p}_x + m^2\omega_c^2 y^2\right) - q\mathcal{E}y, \tag{2.220}$$

thus, the time dependent Schrödinger's equation is written as

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + 2i\frac{m\omega_c}{\hbar}y\frac{\partial\psi}{\partial x} - \frac{m^2\omega_c^2}{\hbar^2}y^2\psi\right) - q\mathcal{E}y\psi. \tag{2.221}$$

Even though if at this point the reader could be tempted to separate the temporal variable, t, and the spatial variable z, via a solution of the form

$$\psi = \psi(x, y, z, t) \quad \rightarrow \quad \psi(x, y, z, t) = \psi_1(x, y)\psi_2(z)\psi_3(t) \tag{2.222}$$

it turns out, ironically, that the eigenvalue equation that comes with this process is more complicated to solve than the time dependent equation. Instead of going through this process, we are going to solve the full Schrödinger's equation. First, is necessary to propose the following time dependent form of solution

$$\psi = \psi(x,y,z,t) \quad \rightarrow \quad \psi(x,y,z,t) = \psi_1(x,y,t)\psi_2(z,t) \tag{2.223}$$

which will give us the couple of time dependent Schrödinger's equation

$$i\hbar\frac{\partial\psi_1}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial y^2} + 2i\frac{m\omega_c}{\hbar}y\frac{\partial\psi_1}{\partial x} - \frac{m^2\omega_c^2}{\hbar^2}y^2\psi_1\right) - q\mathcal{E}y\psi_1. \tag{2.224}$$

$$i\hbar\frac{\partial\psi_2}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi_2}{\partial z^2}, \tag{2.225}$$

then, is not difficult to figure out that the solution for the partial differential equation (2.224) is given by

$$\psi_2(z,t) = \frac{1}{\sqrt{L_z}}\exp\left(i\frac{\sqrt{2mE_z}}{\hbar}z - i\frac{E_z}{\hbar}t\right), \tag{2.226}$$

where E_z is a real constant. Similarly as in the prior cases, if we use Born normalization, we get that

$$E_z = E_{n'} = \frac{1}{2m}\left(\frac{2\pi\hbar}{L_z}n'\right)^2, \quad n' \in \mathbb{Z}^+. \tag{2.227}$$

To solve the equation (2.224) we begin by performing the Fourier transform respect the variable x, defined by eq.(2.30), defining

$$\mathcal{F}_x(\psi_1) = \bar{\psi}(\kappa,y,z,t) \tag{2.228}$$

and using the property (A.4) and rearranging we can write the equation in the Fourier space

$$i\hbar\frac{\partial\bar{\psi}}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\bar{\psi}}{\partial y^2} - \frac{m^2\omega_c^2}{\hbar^2}y^2\bar{\psi} + 2\frac{m\omega_c}{\hbar}\left(\kappa + \frac{q\mathcal{E}}{\hbar\omega_c}\right)y\bar{\psi} - \kappa^2\bar{\psi}\right), \tag{2.229}$$

now, we can complete the squared to rewrite it as follows

$$i\hbar\frac{\partial\bar{\psi}}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\bar{\psi}}{\partial y^2} - \left(\frac{m\omega_c}{\hbar}y - \kappa - \frac{q\mathcal{E}}{\hbar\omega_c}\right)^2\bar{\psi} + \left(\kappa + \frac{q\mathcal{E}}{\hbar\omega_c}\right)^2\bar{\psi} - \kappa^2\bar{\psi}\right), \tag{2.230}$$

which can be rewritten as

$$i\hbar\frac{\partial\bar{\psi}}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\bar{\psi}}{\partial y^2} - \left(\frac{m\omega_c}{\hbar}y - \kappa - \frac{q\mathcal{E}}{\hbar\omega_c}\right)^2\bar{\psi} + 2\frac{q\mathcal{E}}{\hbar\omega_c}\kappa\bar{\psi} + \left(\frac{q\mathcal{E}}{\hbar\omega_c}\right)^2\bar{\psi}\right). \tag{2.231}$$

Now, we propose the following solution in the Fourier space

$$\bar{\psi}(\kappa, y, t) = e^{-i\frac{E}{\hbar}t}\varphi(\kappa, y), \tag{2.232}$$

substituting this and rearranging gives the eigenvalue equation in Fourier space

$$\left(E + \frac{q\mathcal{E}\hbar}{m\omega_c}\kappa + \frac{1}{2m}\left(\frac{q\mathcal{E}}{\omega_c}\right)^2\right)\varphi = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\varphi}{\partial y^2} - \frac{m^2\omega_c^2}{\hbar^2}\left(y - \frac{\hbar}{m\omega_c}\kappa - \frac{q\mathcal{E}}{m\omega_c^2}\right)^2\varphi\right), \tag{2.233}$$

thus, we got, again, the displaced harmonic oscillator equation in the Fourier space. We continue making the change of variable

$$\xi = \sqrt{\frac{m\omega_c}{\hbar}}\left(y - \frac{\hbar}{m\omega_c}\kappa - \frac{q\mathcal{E}}{m\omega_c^2}\right), \tag{2.234}$$

which changes the differential operator as

$$\frac{\partial^2}{\partial y^2} = \frac{m\omega_c}{\hbar}\frac{\partial^2}{\partial \xi^2}, \tag{2.235}$$

and defining

$$E' = E + \frac{q\mathcal{E}\hbar}{m\omega_c}\kappa + \frac{1}{2m}\left(\frac{q\mathcal{E}}{\omega_c}\right)^2, \tag{2.236}$$

then, the differential equation takes the form

$$-\frac{2E'}{\hbar\omega_c}\varphi = \frac{\partial^2\varphi}{\partial y^2} - \xi^2\varphi. \tag{2.237}$$

Therefore, the solution is given by

$$\varphi_n(\xi) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\xi^2}{2}\right)H_n(\xi), \tag{2.238}$$

and the eigenvalues are

$$E' = E_n = \hbar\omega_c\left(n + \frac{1}{2}\right), \quad \text{then} \quad E = \hbar\omega_c\left(n + \frac{1}{2}\right) - \frac{q\mathcal{E}\hbar}{m\omega_c}\kappa - \frac{1}{2m}\left(\frac{q\mathcal{E}}{\omega_c}\right)^2 \tag{2.239}$$

Hence, the solution in the Fourier space is

$$\bar{\psi}(\kappa, y, t) = \exp\left[-i\left(\hbar\omega_c\left(n + \frac{1}{2}\right) - \frac{q\mathcal{E}\hbar}{m\omega_c}\kappa - \frac{1}{2m}\left(\frac{q\mathcal{E}}{\omega_c}\right)^2\right)\frac{t}{\hbar}\right]\varphi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left(y - \frac{\hbar}{m\omega_c}\kappa - \frac{q\mathcal{E}}{m\omega_c^2}\right)\right). \tag{2.240}$$

Now, it is necessary to take the inverse Fourier transform of the above solution. Lets recall the definition of the inverse Fourier transform which is as follows

$$\mathcal{F}_\kappa^{-1}(\phi(\kappa)) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\kappa x}\phi(\kappa)d\kappa. \tag{2.241}$$

Then,

$$\psi_1(x,y,t) = \mathcal{F}_\kappa^{-1}(\bar{\psi}) = \exp\left[-i\left(\hbar\omega_c\left(n+\frac{1}{2}\right) - \frac{1}{2m}\left(\frac{q\mathcal{E}}{\omega_c}\right)^2\right)\frac{t}{\hbar}\right] \times$$
$$\times \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left(-i\left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)\kappa\right)\varphi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left(y - \frac{\hbar}{m\omega_c}\kappa - \frac{q\mathcal{E}}{m\omega_c^2}\right)\right)d\kappa, \tag{2.242}$$

now, changing the variable

$$\kappa = \frac{m\omega_c}{\hbar}y - \frac{q\mathcal{E}}{\hbar\omega_c} - \sqrt{\frac{m\omega_c}{\hbar}}\xi, \tag{2.243}$$

we can write

$$\psi_1(x,y,t) = \exp\left[-i\left(\hbar\omega_c\left(n+\frac{1}{2}\right) - \frac{1}{2m}\left(\frac{q\mathcal{E}}{\omega_c}\right)^2\right)\frac{t}{\hbar} - i\left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)\left(\frac{m\omega_c}{\hbar}y - \frac{q\mathcal{E}}{\hbar\omega_c}\right)\right] \times$$
$$\times\left(-\sqrt{\frac{m\omega_c}{\hbar}}\right)\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left(i\sqrt{\frac{m\omega_c}{\hbar}}\left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)\xi\right)\varphi_n(\xi)d\xi, \tag{2.244}$$

hence, we realize that, similarly as in the case where we described the magnetic field using the Landau's gauge, the final integral is the Fourier transform of an harmonic oscillator, thus, using the result of the appendix (A.1) which implies that the Fourier transform of a harmonic oscillator is another harmonic oscillator, we can write the solution of the time dependent Schrödinger equation as

$$\psi_n(x,y,t) = A\exp\left(-i\chi_n(x,y,t)\right)\phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)\right), \tag{2.245}$$

where A is a normalization constant, the definition $\psi_1 = \psi_n$ was made, the phase was defined as follows

$$\chi_n(x,y,t) = \left(\hbar\omega_c\left(n+\frac{1}{2}\right) - \frac{1}{2m}\left(\frac{q\mathcal{E}}{\omega_c}\right)^2\right)\frac{t}{\hbar} + \left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)\left(\frac{m\omega_c}{\hbar}y - \frac{q\mathcal{E}}{\hbar\omega_c}\right) \tag{2.246}$$

and the harmonic oscillator function was denoted by

$$\phi_n(\xi) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega_c}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\xi^2}{2}\right)H_n(\xi). \tag{2.247}$$

Similarly than in the case without electric field, that is $\mathcal{E} = 0$, the normalization constant can be calculated with the expression

$$\int_{-L_y/2}^{L_y/2}\int_{-\infty}^{\infty} |\psi_n(x,y,t)|^2 dxdy = 1, \tag{2.248}$$

and is not difficult to find that $A = 1/\sqrt{L_y}$.
To end with this section, one must recall that the expression eq.(2.245) satisfies the two dimensional time dependent Schrödinger's equation, given by (2.224), instead of the eigenvalue equation. Due to the importance of this solution, a proof of this last statment is given in the appendix (C.1).

2.3.1 Degeneration of the system with electromagnetic field

To study the degeneration of the system we are going to focus in the two dimensional equation (2.224) which define us the Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 + 2m\omega_c y\hat{p}_x + m^2\omega_c^2 y^2\right) - q\mathcal{E}y, \tag{2.249}$$

as the reader could note, the selection of the electric field potential has no effect with the conservation of the linear momentum in x direction as in the case where $\mathcal{E} = 0$, that is,

$$[\hat{p}_x, \hat{\mathcal{H}}] = 0, \tag{2.250}$$

the operator $\hat{p}_x$ stills commutes with the Hamiltonian. Therefore, since the application of this operator do not gives a function proportional to the original one, $\hat{p}_x\psi_n \not\sim \psi_n$, we can expect to have a set of solutions of the time dependent Scrödinger's equation. This it easy to prove, we can write the generalization of the above commutation argument as

$$[\hat{p}_x^j, \hat{\mathcal{H}}] = 0, \qquad j \in \mathbb{Z}^+, \tag{2.251}$$

where the index j defines the j-th application of the operator. Then, being the solution eq.(2.245) such that

$$\hat{\mathcal{H}}\psi_n = i\hbar\frac{\partial\psi_n}{\partial t}, \tag{2.252}$$

from the commutation relation (2.251) applied to this solution we have that

$$\hat{p}_x^j\left(\hat{\mathcal{H}}\psi_n\right) = \hat{\mathcal{H}}\left(\hat{p}_x^j\psi_n\right), \tag{2.253}$$

hence

$$i\hbar\frac{\partial}{\partial t}\left(\hat{p}_x^j\psi_n\right) = \hat{\mathcal{H}}\left(\hat{p}_x^j\psi_n\right). \tag{2.254}$$

Therefore, the function $\hat{p}_x^j\psi_n$ is also a solution of the complete Scrödinger's equation. Similarly as we did in the section (2.1.1) is posible to find an expression for the j-th application of the momentum operator, defining the functions as

$$f_n^j(x,y,t) = \hat{p}_x^j\psi_n(x,y,t), \qquad j = 0,1,2,... \tag{2.255}$$

using the equalities (2.62) and (2.82) one can demonstrate that the eigenfunctions has the following form

$$f_n^{j+1}(x,y,t) = \left[m\omega_c\left(i\left(x - \frac{q\mathcal{E}}{m\omega_c}t\right) - y\right) + \frac{q\mathcal{E}}{\omega_c}\right]f_n^j + m\omega_c\hbar j f_n^{j-1} - i\sqrt{2nm\omega_c\hbar}f_{n-1}^j. \tag{2.256}$$

At this point, one could try, again, to invert the magnetic field direction and use the definitions of the momentum operators (2.14) and (2.15) to try to obtain a couple of operators that commutes with the Hamiltonian. However, doing this process using the gauge $\mathbf{A}' = B(0,-x,0)$, we obtain the operators

$$\hat{\pi}'_x = \hat{p}_x, \quad \hat{\pi}'_y = \hat{p}_y + m\omega_c x, \tag{2.257}$$

and is a straightforward calculation to prove that

$$[\hat{\pi}'_y, \hat{\mathcal{H}}] = i\hbar q\mathcal{E}, \tag{2.258}$$

hence, the operator $\hat{\pi}'_y$ is not conserved anymore. Regardless this situation, is still possible to find a second eigenvalue equation parametrized for the degeneration index j. First, we calculate the action of the momentum $\hat{p}_y$ over the function eq.(2.245)

$$\hat{p}_y\psi_n = -m\omega_c\left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)\psi_n. \tag{2.259}$$

Second, we apply j times the momentum operator $\hat{p}_x$ to the above expression

$$(\hat{p}_x)^j\hat{p}_y\psi_n = -m\omega_c(\hat{p}_x)^j\left[\left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)\psi_n\right]. \tag{2.260}$$

Using the fact that $[(\hat{p}_x)^j, \hat{p}_y] = 0$ and the differentiation of two functions, eq.(2.82), we can write

$$\hat{p}_y f_n^j = -m\omega_c\left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)f_n^j + im\omega_c\hbar j f_n^{j-1}, \tag{2.261}$$

doing some rearrangement, changing $j \to j+1$ and using the definitions (2.257), the prior can be rewritten as

$$(\hat{\pi}'_y - q\mathcal{E}t)\hat{\pi}'_x f_n^j = im\omega_c\hbar(j+1)f_n^j. \tag{2.262}$$

And the operator

$$\hat{\Pi}'_y = \hat{\pi}'_y - q\mathcal{E}t, \tag{2.263}$$

is conserved, as can be corroborated by using the evolution of an operator in Heisenberg scheme

$$\frac{d\hat{f}}{dt} = \frac{1}{i\hbar}[\hat{f}, \hat{H}] + \frac{\partial\hat{f}}{\partial t}. \tag{2.264}$$

There is a third conserved operator which is the the energy operator defined as

$$\hat{E} = i\hbar\frac{\partial}{\partial t}, \tag{2.265}$$

do not share bases with the Hamiltonian, instead it is a generator of solutions of the complete Schrödinger equation. This is easy to prove, using the fact that any power of this operator commutes with the Hamiltonian, that is, $[(\hat{E})^j, \hat{H}] = 0$, where $j \in \mathbb{Z}^+$, then it follows that

$$(\hat{E})^j\hat{H}\psi_n = \hat{H}(\hat{E})^j\psi_n, \tag{2.266}$$

defining the functions $g_n^j(x, y, t) = (\hat{E})^j\psi_n(x, y, t)$ and using the expression eq.(D.55), it follows that

$$i\hbar\frac{\partial g_n^j}{\partial t} = \hat{H}g_n^j. \tag{2.267}$$

Hence, the energy operator combined with the operator eq.(2.263) can be used to find a second eigenvalue expression

$$\hat{\Pi}'_y\hat{E}g_n^j = iq\mathcal{E}\hbar(j+1)g_n^j. \tag{2.268}$$

To end up with this section, we must say that a couple of extra cases dealing with electromagnetic field are solved in the appendix (C).

2.3.2 Summary of the results with electromagnetic field

Finally, to finish this chapter we present a summary of this last case results. Having an electromagnetic field, such that

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{2.269}$$

and

$$\mathbf{E} = -\nabla \phi. \tag{2.270}$$

where $\mathbf{A}$ and ϕ are the vector potential and the electric potential respectively and such that they are perpendicular, $\mathbf{B} \perp \mathbf{E}$, we use Landau's gauge with magnetic field intensity of $|\mathbf{B}| = B$

$$\mathbf{A} = B(-y, 0, 0), \tag{2.271}$$

and select the electric field potential as

$$\phi = -\mathcal{E}y, \tag{2.272}$$

such that the electric field intensity is $|\mathbf{E}| = \mathcal{E}$. This define us the following Hamiltonian

$$\hat{\mathbf{H}} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 + 2m\omega_c y \hat{p}_x + m^2\omega_c^2 y^2\right) - q\mathcal{E}y, \tag{2.273}$$

where the following operators are conserved

$$\hat{\pi}'_x = \hat{p}_x, \tag{2.274}$$

$$\hat{\Pi}'_y = \hat{\pi}'_y - q\mathcal{E}t, \tag{2.275}$$

$$\hat{E} = i\hbar\frac{\partial}{\partial t}. \tag{2.276}$$

The later define us the solutions of the time dependent Schrödinger equations as

$$\psi_n^j(x, y, z, t) = f_n^j(x, y, t)\frac{1}{\sqrt{L_z}}\exp\left(i\frac{2\pi}{L_z}n'z - i\frac{E_{n'}}{\hbar}t\right), \tag{2.277}$$

where

$$E_{n'} = \frac{1}{2m}\left(\frac{2\pi\hbar}{L_z}n'\right)^2, \quad n' \in \mathbb{Z}^+. \tag{2.278}$$

and the following definitions where made,

$$f_n^j(x, y, t) = \hat{p}_x^j\psi_n(x, y, t), \qquad j = 0, 1, 2, ... \tag{2.279}$$

with

$$\psi_n(x, y, t) = \frac{1}{\sqrt{L_y}}\exp\left(-i\chi_n(x, y, t)\right)\phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)\right), \tag{2.280}$$

the phase is define as

$$\chi_n(x, y, t) = \left(\hbar\omega_c\left(n + \frac{1}{2}\right) - \frac{1}{2m}\left(\frac{q\mathcal{E}}{\omega_c}\right)^2\right)\frac{t}{\hbar} + \left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)\left(\frac{m\omega_c}{\hbar}y - \frac{q\mathcal{E}}{\hbar\omega_c}\right), \tag{2.281}$$

and the harmonic oscillator function was denoted by

$$\phi_n(\xi) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega_c}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{\xi^2}{2}\right) H_n(\xi). \tag{2.282}$$

Then we have that the functions eq.(2.277) satisfies the equalities

$$\hat{\mathbf{H}} f_n^j = i\hbar \frac{\partial f_n^j}{\partial t}, \tag{2.283}$$

and, defining the canonic momentum operators obtained by the inversion of the magnetic field direction eq.(2.257),

$$\hat{\pi}'_x = \hat{p}_x, \quad \hat{\pi}'_y = \hat{p}_y + m\omega_c x, \tag{2.284}$$

we have that

$$(\hat{\pi}'_y - q\mathcal{E}t)\hat{\pi}'_x \psi_n^j = im\omega_c\hbar(j+1)\psi_n^j. \tag{2.285}$$

Also, using the energy operator, we can define a new set of solutions for the time dependent Schrödinger equation. Defining

$$g_n^j(x,y,t) = \hat{E}^j \psi_n(x,y,t), \qquad j = 0,1,2,... \tag{2.286}$$

were the solutions are written as

$$g_n^{j+1}(x,y,t) = \left(\alpha_n(y) + \frac{q\mathcal{E}}{\hbar}\left(x - \frac{q\mathcal{E}}{m\omega_c}\right)\right) g_n^j - ij\frac{q^2\mathcal{E}}{m\omega_c} g_n^{j-1} - \frac{q\mathcal{E}}{m\omega_c}\sqrt{2n\frac{m\omega_c}{\hbar}} e^{-i\hbar\omega_c} g_{n-1}^j, \tag{2.287}$$

and the following definition was made

$$\alpha_n(y) = \hbar\omega_c\left(n + \frac{1}{2}\right) - \frac{1}{2m}\left(\frac{q\mathcal{E}}{\omega_c}\right)^2 + \left(\frac{m\omega_c}{\hbar} y - \frac{q\mathcal{E}}{\hbar\omega_c}\right), \tag{2.288}$$

we have that all the g_n^j functions satisfies the expressions

$$\hat{\mathbf{H}} g_n^j = i\hbar \frac{\partial g_n^j}{\partial t}, \tag{2.289}$$

and also satisfies the eigenvalue equation

$$\hat{\Pi}'_y \hat{E} g_n^j = iq\mathcal{E}\hbar(j+1) g_n^j. \tag{2.290}$$

Hence, the general solution can be written as a linear combination of the basis, that is

$$\Psi(x,y,z,t) = \sum_{n,n',j,j'} C_{n,n',j,j'} \hat{E}^{j'} \hat{p}_x^j \psi_n(x,y,t) \frac{1}{\sqrt{L_z}} \exp\left(i\frac{2\pi}{L_z} n' z - i\frac{E_{n'}}{\hbar} t\right), \tag{2.291}$$

where $C_{n,n',j,j'}$ are complex constants.

Chapter 3

Analogous quantum Hall effect

3.1 Definition of the current

In the previous chapter, we have shown how to work with Hamiltonians involving electromagnetic fields, therefore, we already have our wave functions to work with. However, there is still a definition we need and is that one of electric current which is going to be deduced in this section. We beging with the no relativistic Hamiltonian (2.6)

$$\hat{\mathbf{H}} = \frac{1}{2m}\left(\hat{\mathbf{P}} - \frac{q}{c}\mathbf{A}\right)^2 + V, \tag{3.1}$$

where $V \in \mathbb{R}$ is a scalar function and the vector field with domain and image in the reals $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$, then, we substitute the momentum operator $\hat{\mathbf{P}} = -i\hbar\nabla$ and expanding the squared term to write

$$\hat{\mathbf{H}} = \frac{1}{2m}\left(-\hbar^2\nabla^2 + i\frac{\hbar q}{c}(\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla) + \frac{q^2}{c^2}\mathbf{A}^2\right) + V, \tag{3.2}$$

therefore, Schrödinger's equation is written as

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(-\hbar^2\nabla^2\psi + i\frac{\hbar q}{c}(\nabla\cdot(\mathbf{A}\psi) + \mathbf{A}\cdot\nabla\psi) + \frac{q^2}{c^2}\mathbf{A}^2\psi\right) + V\psi. \tag{3.3}$$

We can take the complex conjugate of this last expression to write

$$-i\hbar\frac{\partial\psi^*}{\partial t} = \frac{1}{2m}\left(-\hbar^2\nabla^2\psi^* - i\frac{\hbar q}{c}(\nabla\cdot(\mathbf{A}\psi^*) + \mathbf{A}\cdot\nabla\psi^*) + \frac{q^2}{c^2}\mathbf{A}^2\psi^*\right) + V\psi^*, \tag{3.4}$$

multiplying eq.(3.3) by ψ^* and eq.(3.4) by $-\psi$, adding up both results and making some rearrangements we can write

$$i\hbar\frac{\partial(\psi\psi^*)}{\partial t} = \frac{1}{2m}\left(\hbar^2(\psi\nabla^2\psi^* - \psi^*\nabla^2\psi) + i\frac{\hbar q}{c}\left(\psi^*\nabla\cdot(\mathbf{A}\psi) + \psi^*\mathbf{A}\cdot\nabla\psi + \psi\nabla\cdot(\mathbf{A}\psi^*) + \psi\mathbf{A}\cdot\nabla\psi^*\right)\right) \tag{3.5}$$

realizing that

$$\nabla\cdot(\psi\nabla\psi^* - \psi^*\nabla\psi) = \psi\nabla^2\psi^* - \psi^*\nabla^2\psi, \tag{3.6}$$

and also

$$\psi^* \nabla \cdot (\mathbf{A}\psi) + \psi^* \mathbf{A} \cdot \nabla \psi + \psi \nabla \cdot (\mathbf{A}\psi^*) + \psi \mathbf{A} \cdot (\nabla \psi^*) = 2\nabla \cdot (\mathbf{A}\psi^* \psi), \tag{3.7}$$

substituting this two equalities, simplifying and rearranging it we can write the following expression

$$\frac{\partial(\psi\psi^*)}{\partial t} + \nabla \cdot \left(\frac{i\hbar}{2m} (\psi \nabla \psi^* - \psi^* \nabla \psi) - \frac{q}{mc} \mathbf{A}\psi^* \psi \right) = 0. \tag{3.8}$$

The above expression is nothing but the continuity equation without sources, which is written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \tag{3.9}$$

where

$$\rho = \psi\psi^*, \tag{3.10}$$

represent the particle probability density function and the current is define as

$$\mathbf{J} = \frac{i\hbar}{2m} (\psi \nabla \psi^* - \psi^* \nabla \psi) - \frac{q}{mc} \mathbf{A}\psi^* \psi. \tag{3.11}$$

Note that the above current definition could be written as $\mathbf{J} = \mathbf{v}\rho$ where $\mathbf{v}$ is the particle velocity, since we are interested in the electrical current is necessary to multiply it times the particle charge q, therefore, the electrical current is $\mathbf{J}_e = q\mathbf{J}$ so

$$\mathbf{J}_e = \frac{iq\hbar}{2m} (\psi \nabla \psi^* - \psi^* \nabla \psi) - \frac{q^2}{mc} \mathbf{A}\psi^* \psi. \tag{3.12}$$

A useful alternative form of the above equality can be deduced if we write the wave function in polar coordinates. Since ψ is a complex function $\psi : \mathbb{C} \to \mathbb{C}$, therefore, we can write it as

$$\psi = Re(\psi) + iIm(\psi) = \sqrt{Re(\psi)^2 + Im(\psi)^2} e^{i \arctan\left(\frac{Im(\psi)}{Re(\psi)}\right)}, \tag{3.13}$$

or, since $\rho = |\psi|^2 = Re(\psi)^2 + Im(\psi)^2$ the radius can be define as $r = \sqrt{\rho}$ and $\theta = \arctan\left(\frac{Im(\psi)}{Re(\psi)}\right)$, then

$$\psi = re^{i\theta}, \tag{3.14}$$

substituting it in eq.(3.17) using the following diferentiation results

$$\nabla \psi = (\nabla r + ir\nabla\theta) e^{i\theta}, \tag{3.15}$$

and

$$\psi \nabla \psi^* - \psi^* \nabla \psi = -2ir^2 \nabla \theta, \tag{3.16}$$

we can write the electrical current in therms of the phase as

$$\mathbf{J}_e = \left(\frac{q\hbar}{m} \nabla \theta - \frac{q^2}{mc} \mathbf{A} \right) |\psi|^2. \tag{3.17}$$

3.2 Single charged particle electrical current

This section is strongly based on the results obtained in the appendix (D). As we mention before in appendix (D.2.3), we are going to deduce an analogous phenomena to the quantum Hall effect using the solution (D.73), since the quantum Hall effect is a phenomena that appears when a electromagnetic field is presented. However, since in this particular phenomena the electron gas is constrained to a two dimensional surface whose normal vector is parallel to the magnetic field, we only need the $x - y$ solution of the respective wave function, that is

$$\Psi(x, y, t) = \sum_n C_n \psi_n(x - \delta x, y, t - \delta t), \tag{3.18}$$

where

$$\psi_n(x, y, t) = \frac{1}{\sqrt{L_y}} \exp\left(-i\chi_n(x, y, t)\right) \phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)\right), \tag{3.19}$$

the phase is define as

$$\chi_n(x, y, t) = \left(\hbar\omega_c\left(n + \frac{1}{2}\right) - \frac{1}{2m}\left(\frac{q\mathcal{E}}{\omega_c}\right)^2\right)\frac{t}{\hbar} + \left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)\left(\frac{m\omega_c}{\hbar}y - \frac{q\mathcal{E}}{\hbar\omega_c}\right), \tag{3.20}$$

and the harmonic oscillator function was denoted by

$$\phi_n(\xi) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega_c}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{\xi^2}{2}\right) H_n(\xi). \tag{3.21}$$

A experimental feature about the QHE is that the temperature is decreased at the point that the systems is found near the lowest Landau's level, therefore, the wave function has the following form

$$\Psi(x, y, t) = \frac{C_0}{\sqrt{L_y}} \exp\left(-i\chi_0(x - \delta x, y, t - \delta t)\right) \phi_0\left(\sqrt{\frac{m\omega_c}{\hbar}}\left(x - \delta x - \frac{q\mathcal{E}}{m\omega_c}t - \delta t\right)\right), \tag{3.22}$$

hence, since the harmonic oscillator is a function with domain and image in the reals and the constant C_0 can be written as

$$C_0 = \sqrt{Re(C_0)^2 + Im(C_0)^2}\, e^{i\arctan\left(\frac{Im(C_0)}{Re(C_0)}\right)}, \tag{3.23}$$

making the definition $\Delta x = x - \delta x$ and $\Delta t = t - \delta t$, the phase is equal to

$$\theta = -\frac{\hbar\omega_c\Delta t}{2\hbar} + \frac{1}{2m}\left(\frac{q\mathcal{E}}{\omega_c}\right)^2\frac{\Delta t}{\hbar} - \left(\Delta x - \frac{q\mathcal{E}}{m\omega_c}\Delta t\right)\left(\frac{m\omega_c}{\hbar}y - \frac{q\mathcal{E}}{\hbar\omega_c}\right) + \arctan\left(\frac{Im(C_0)}{Re(C_0)}\right). \tag{3.24}$$

Now, we substitute it in the definition of the electrical current eq.(3.17) having calculated the phase gradient to be

$$\nabla\theta = -\left(\frac{m\omega_c}{\hbar}y - \frac{q\mathcal{E}}{\hbar\omega_c}\right)\hat{\imath} - \frac{m\omega_c}{\hbar}\left(\Delta x - \frac{q\mathcal{E}}{m\omega_c}\Delta t\right)\hat{\jmath}, \tag{3.25}$$

and knowing that the gauge $\mathbf{A} = B(-y, 0, 0)$ was used, after simplifying, we got the electrical current to be

$$\mathbf{J}_e = \left(\frac{qc\mathcal{E}}{B}\hat{\imath} - q\omega_c\left(\Delta x - \frac{q\mathcal{E}}{m\omega_c}\Delta t\right)\hat{\jmath}\right)|\Psi|^2. \tag{3.26}$$

Since the electrical current definition is $\mathbf{J}_e = \sigma \mathbf{E}$, therefore, where the electric field for this system is $\mathbf{E} = (0, \mathcal{E}, 0)$ and the conductivity is a 3×3 matrix

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}, \tag{3.27}$$

then, the electrical conductivity can be written as

$$\mathbf{J}_e = \mathcal{E}(\sigma_{xy}\hat{\imath} + \sigma_{yy}\hat{\jmath} + \sigma_{zy}\hat{k}), \tag{3.28}$$

and by association with the conductivities eq.(3.26) are

$$\sigma_{xy} = \frac{qc}{B}|\Psi|^2, \qquad \sigma_{yy} = -\frac{q\omega_c}{\mathcal{E}}\left(\Delta x - \frac{q\mathcal{E}}{m\omega_c}\Delta t\right)|\Psi|^2, \qquad \sigma_{zy} = 0. \tag{3.29}$$

Lets remember that the resistivity is the inverse of the conductivity and, due to its direction, σ_{xy} is the Hall conductivity, then, we are going to denote the Hall resistivity as

$$\rho_H = \frac{B}{qc}\frac{1}{|\Psi|^2}, \tag{3.30}$$

and the longitudinal resistivity is define as

$$\rho_L = -\frac{\mathcal{E}}{q\omega_c\left(\Delta x - \frac{q\mathcal{E}}{m\omega_c}\Delta t\right)}\frac{1}{|\Psi|^2}, \tag{3.31}$$

the resistivity along z direction is meaningless since the system is constrained to a plane and there is no movement in that direction. In QHE experiments, what is measured is resistivity [3, 22], therefore, our observable quantity is the mean value of the above resistivity definitions. The expected value of the Hall resistivity eq.(3.30) is

$$\langle\Psi|\,\rho_H\,|\Psi\rangle = \frac{B}{qc}\iint\limits_{\Omega}\frac{|\Psi|^2}{|\Psi|^2}\,dx\,dy, \tag{3.32}$$

where Ω is the integration domain which, in this case, is limited to a plane of area $A = L_xL_y$, then, multiplying and dividing the above equality by $m\omega_c/\hbar$ on the right hand side and substituting the value of $\omega_c = qB/mc$ in the denominator we can write down the Hall resistivity as

$$\langle\Psi|\,\rho_H\,|\Psi\rangle = \frac{\hbar}{q^2}\left(\frac{m\omega_c}{\hbar}A\right). \tag{3.33}$$

On the other hand, the expected value of the longitudinal resistivity, eq.(3.31), is

$$\rho_L = -\iint\limits_{\Omega}\frac{\mathcal{E}}{q\omega_c\left(\Delta x - \frac{q\mathcal{E}}{m\omega_c}\Delta t\right)}\,dx\,dy, \tag{3.34}$$

this resistivity is expected to vanish in the experimental measurements [3, 14, 15, 22–25, 27, 45]. We note that the only way for the above integral to vanish is if the time is larger than the quantity $\frac{m\omega_c}{q\mathcal{E}}\Delta x$, that is

$$\lim_{t\to\infty} \rho_L = 0, \tag{3.35}$$

since the translation δx must be chosen in a way that the wave function is still inside our domain, inside the plane, we have that $0 \leq |x - \delta x| \leq L_x$, therefore, we can expect the longitudinal resistivity to vanish if

$$\Delta t >> \frac{m\omega_c}{q\mathcal{E}} L_x. \tag{3.36}$$

One needs to mention that the concept of a vanishing resistivity after a given time interval can be find in the literature as *relaxation time* [82, p. 498]. Continuing with the analysis of the expression eq.(3.33), in the introduction of this thesis, we discussed how Landau's quantization of magnetic flux, eq.(1.13), was obtained by Landau himself as a periodicity argument and an analysis of the maxima and minima value of the center of the harmonic oscillator. This implies the quantization of the quantity

$$\frac{m\omega_c}{\hbar} A = 2\pi l, \quad l \in \mathbb{Z}, \tag{3.37}$$

and leads to the deduction of the IQHE via a classic deduction of the Hall resistivity, eq.(1.11). The above expression is also known has the magnetic flux quantization. However, the result we got earlier for the Hall resistivity, eq.(3.33), has been obtained only with the quantum electrical current definition and is quite interesting since it implies that the quantization of the magnetic flux give rise to the quantization of the Hall resistivity such that is directly proportional to the Klitzing's constant, eq.(1.6), in other words, if the magnetic flux is quantized, then, it implies the existence of the FQHE. As we have seen in appendix (D) we showed three different posible reasons of the quantization of this quantity. Hence, for the lowest Landau level, $n = 0$, the general solution is $\Psi(x, y, t) = \psi_0(x - \delta x, y, t - \delta t)$ and using the expression eq.(D.75) one can see immediately that the resistivity is quantized in integer multiples of the von Klitzing constant, that is, $\langle \rho_H \rangle = lh/q^2$. One must mention that this quantization has been already measured experimentally in [22].

The expression eq.(3.33) can be rewritten as

$$\left\langle \frac{\mathcal{E}}{j_x} \right\rangle = \frac{\hbar}{q^2} \left(\frac{m\omega_c}{\hbar} A \right) \longrightarrow \frac{h}{q\delta t\delta y} \left\langle \frac{1}{j_x} \right\rangle = \frac{h}{q^2} \left(\frac{m\omega_c}{\hbar} A \right) \frac{\hbar}{q\mathcal{E}} \frac{1}{\delta t\delta y}, \tag{3.38}$$

note that the unit of the following quantity is the ohm

$$\left[\frac{h}{q\delta t\delta y} \left\langle \frac{1}{j_x} \right\rangle \right] = \Omega, \tag{3.39}$$

Therefore, this quantity express the Hall resistance. Finally, substituting the fields quantizations conditions, eq.(D.75) and eq.(D.76), it can be written as

$$R_H = \frac{h}{q^2} \frac{l}{k}, \tag{3.40}$$

thus, the Hall resistance is quantized in rational numbers of the von Klitzing constant.

3.3 Comparison with the experimental data

From equality (3.37) one can find a relation with the magnetic field, that is

$$B = \frac{hc}{qA} l, \quad l \in \mathbb{Z}, \tag{3.41}$$

for the particular case where $l = 1$, from eq.(3.33), one have that the resistance must be

$$\langle\Psi|\,\rho_H\,|\Psi\rangle = \frac{h}{q^2}, \tag{3.42}$$

this value is easy to identify in the experiments. Since the values of all constants involved in the expression (3.41) are known except for the area A, we can deduce that we must find a way to calculate this last constant. For instance, in the reference [22] which the experimental data is shown in fig.(3.1), since the relation from eq.(3.33) describes a linear behavior, we see that the value of the magnetic field when the resistivity has the value of h/q^2 is the one when the lineal red curve cuts the flat with a filling factor of 1, that is

$$B_0 = 5T, \tag{3.43}$$

which can be used to calculate the area using the expression

$$A = \frac{hc}{qB_0}, \tag{3.44}$$

which in this case turns out to be $A \approx 8.27 \times 10^{-4}\mu m^2$. Now is important to note that we will have the following couple of equalities

$$B = B_0 l, \quad \langle\Psi|\,\rho_H\,|\Psi\rangle = \frac{h}{q^2}l, \tag{3.45}$$

which tell us that we could find another quantization condition when $l = 3$ which implies that $B = 15T$ where the resistivity will be $\langle\rho\rangle = 3h/q^2$, which in fact can be seen in the experimental data shown below. A similar process can be performed in the experimental data on the reference [23] which is shown in the fig.(3.2). For this case one can fin that $B_0 = 9.8T$, which corresponds to an area of $A = 4.22 \times 10^{-4}\mu m^2$. The figure (3.3) shows the experimental data reported on reference (3.3), for this case $B_0 = 4.2T$ which leads to an area of $A = 9.85 \times 10^{-4}\mu m^2$. We continue with the fig.(3.4) where $B_0 = 5.3T$ for an area of $A = 7.8 \times 10^{-4}\mu m^2$. Finally, for the fig.(3.5) $B_0 = 12T$ for an area of $A = 3.45 \times 10^{-4}\mu m^2$. Finally, it is interesting to note that all the flats are localized by a modified magnetic field expression $B = B_0 l/k$ where $l, k \in \mathbb{Z}^+$ once the are is determined.

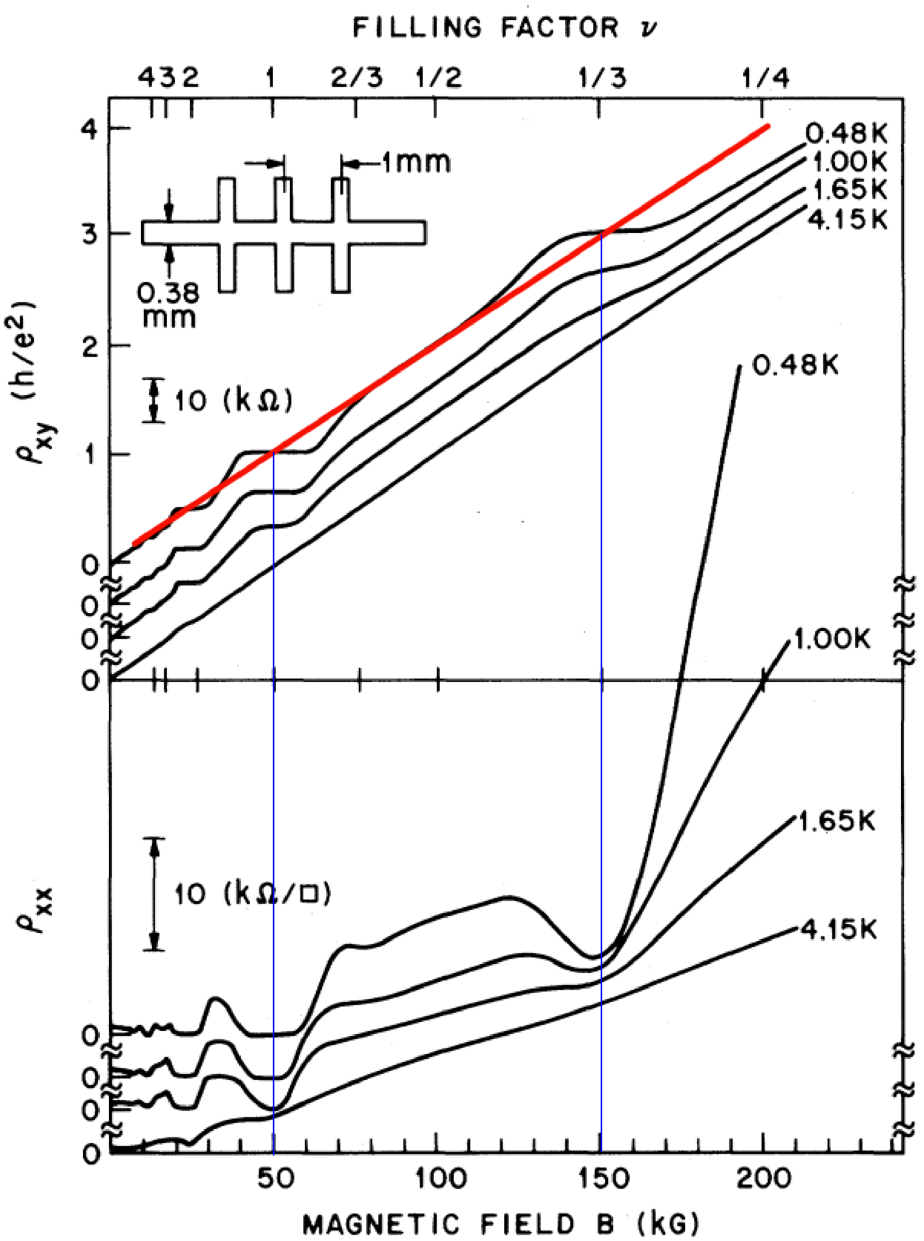

Figure 3.1: Experimental data presented in reference [22]. ρ_{xy} and ρ_{xx} vs B, taken from a GaAs-Al$_{0.3}$-Ga$_{0.7}$As sample with $n = 1.23 \times 10^{11}/cm^2$, $\mu = 90000 cm^2/V$sec, using $I = 1\mu A$. The Landau level filling factor is defined by $\nu = nh/qB$. The red line represents the linear behavior of the resistivity and the blue lines intersects the red line when it crosses the flat plateaus.

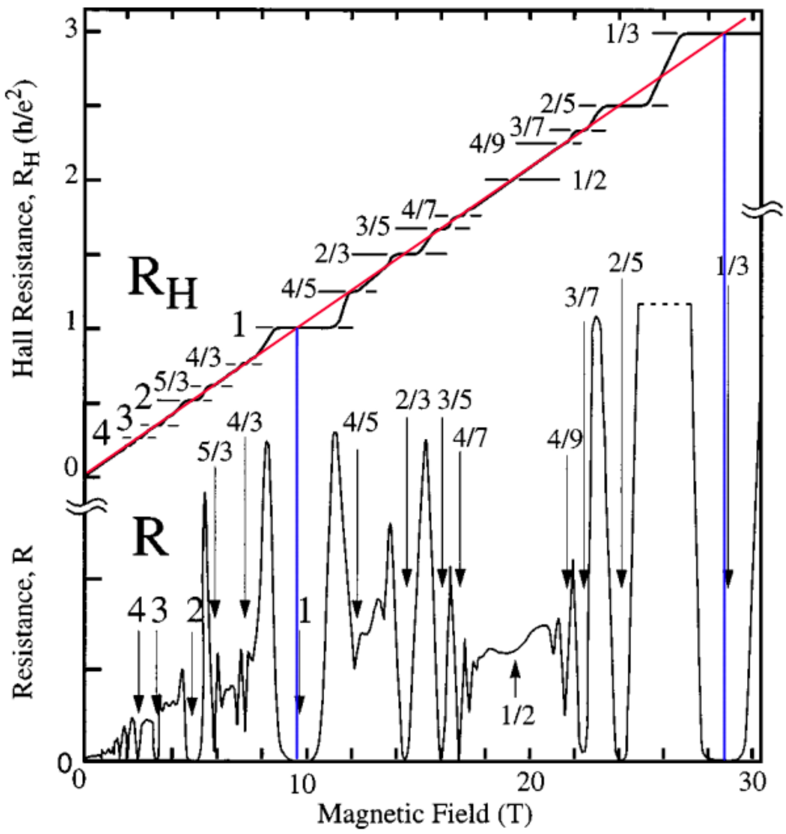

Figure 3.2: Experimental data presented in reference [23]. The FQHE as it appears today in ultrahigh-mobility modulation-doped GaAs/AlGaAs 2DESs. Many fractions are visible. The red line represents the linear behavior of the resistivity and the blue lines intersects the red line when it crosses the flat plateaus.

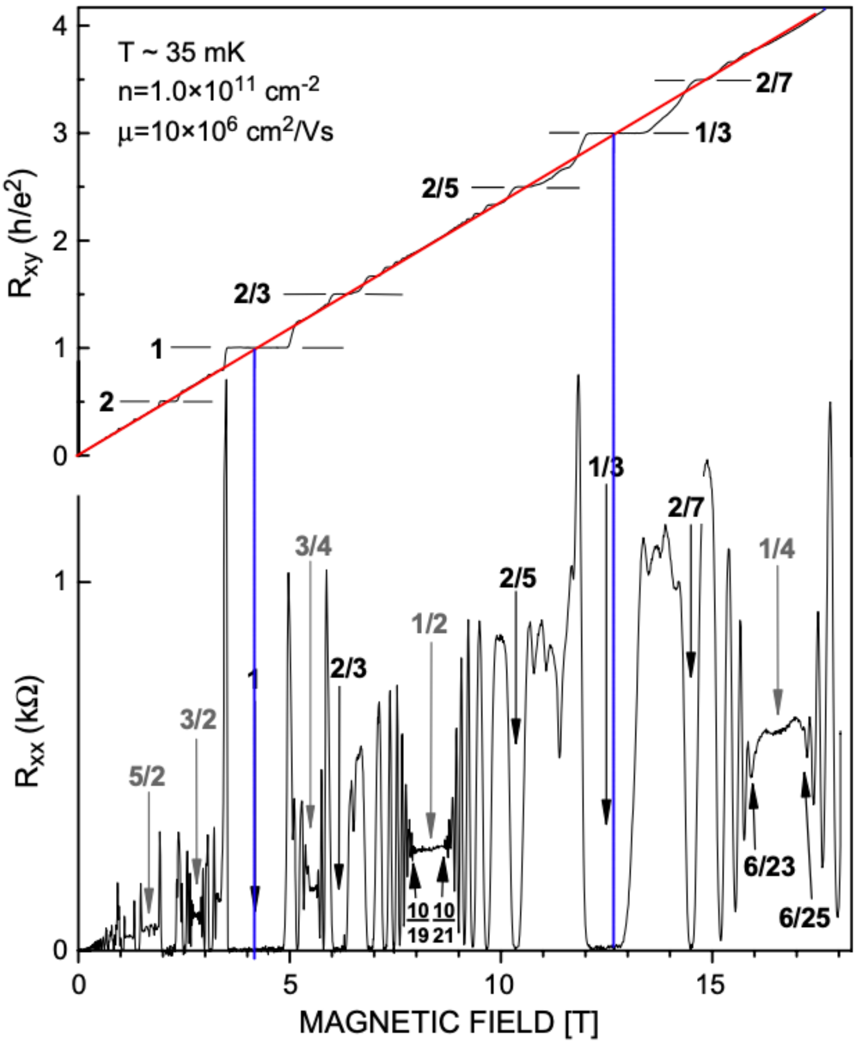

Figure 3.3: Experimental data presented in reference [83]. When the Hall resistance is measured as a function of magnetic field plateaus at quantized values are observed. In regions of the magnetic field where the Hall resistance is in a plateau, the longitudinal resistance vanishes. The red line represents the linear behavior of the resistivity and the blue lines intersects the red line when it crosses the flat plateaus.

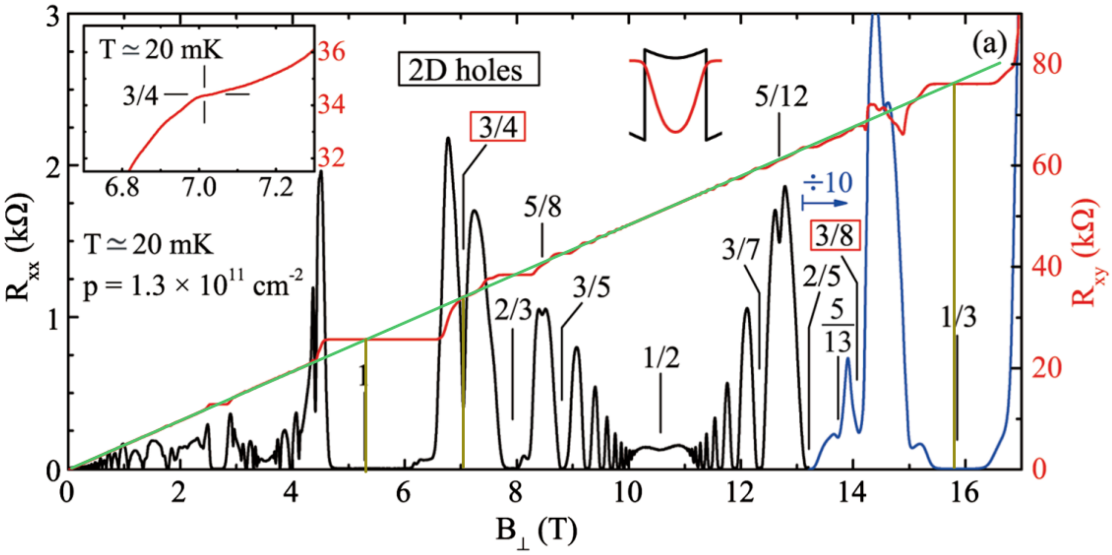

Figure 3.4: Experimental data presented in reference [55]. Longitudinal resistance (R_{xx}, in black and blue) and Hall resistance (R_{xy}, in red) vs perpendicular magnetic field $B_{\perp}$ traces for ultrahigh-mobility 2D hole sample. The height of the Blue trace is divided by a factor of 10. The $B_{\perp}$ positions of severals Landau levels fillings are marked. A strong minimum in R_{xx} accompanied by a developing Hall plateau is observed at $\nu = 3/4$. An enlarged version of the R_{xy} and $B_{\perp}$ near $\nu = 3/4$ at $20mK$ in the top-left inset. The green line represents the linear behavior of the resistivity and the dark-yellow lines intersects the green line when it crosses the flat plateaus.

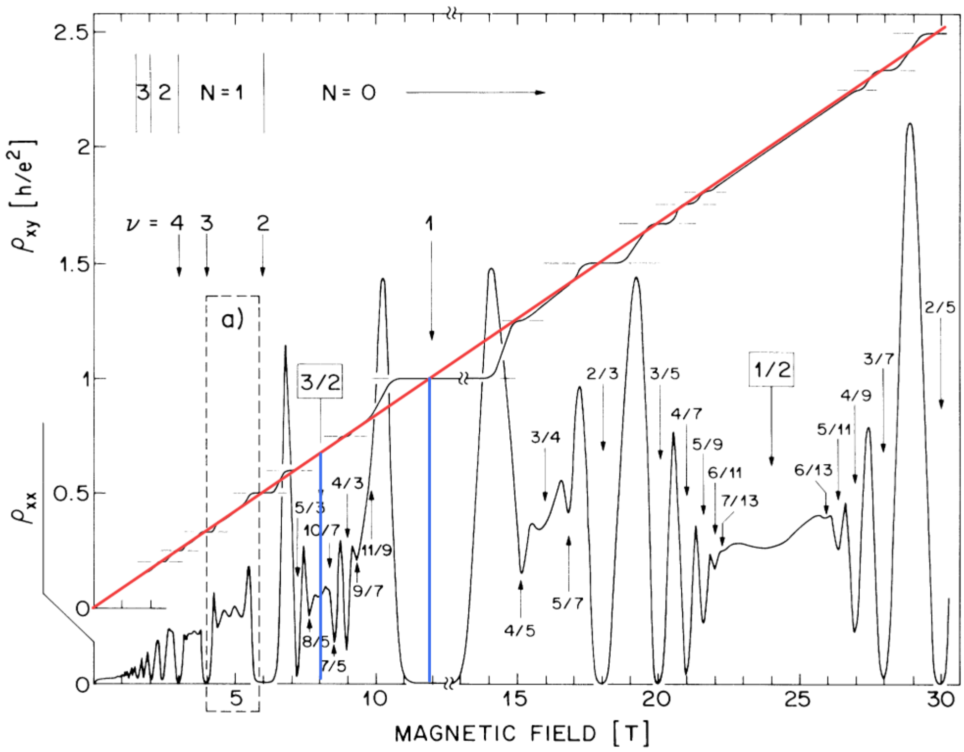

Figure 3.5: Experimental data presented in reference [25]. Overview of diagonal resistivity ρ_{xx} and Hall resistance ρ_{xy} of sample described in the reference. The red line represents the linear behavior of the resistivity and the blue lines intersects the red line when it crosses the flat plateaus.

Conclusions

Through out this work, the non relativistic hamiltonian for electromagnetic field was analyzed using different gauge selection. The results obtained were

a) Non separable variable solutions for all cases were found, that is, using Landau's gauge, symmetric gauge and for electromagnetic field.

b) The degeneration of all the systems were determined obtaining analytical expressions for all the cases. This degeneration is determined by the application of the conserved operators of each system on the solution determined.

c) The energies of the systems were determined being the Landau's level in all cases involving static magnetic field only. For the case with electromagnetic field involved, the time evolution operator do not give an eigenequation solution, instead, it gives more solutions to the time dependent Schrödinger's equation.

d) The symmetries of all the systems were analyzed showing that the electromagnetic flux quantization conditions, eq.(D.76) and (D.75), are necessary if the system is invariant under this unitary transformation. This result was used in the determination of the resistivity of the system showing that if this condition is satisfied then then resistivity is quantized.

e) The above result was compared with the existing experiments at the moment showing that it is related to the phenomena called fractional quantum Hall effect. We used the quantization relation obtained in the appendix (D) and the result of the section (3.2) to calculate the mid point of the quantum hall effect plateaus which were determined successfully. This implies that the fractional quantum Hall effect is an observable manifestation of the wave function invariance under the application of the operator (D.68).

Appendix A

Calculation details for the Landau's gauge

A.1 Fourier transform of a harmonic oscillator

In this section we are going to study some properties of the Fourier transform and apply them to the harmonic oscillator problem. Being the function $\phi = \phi(x)$ belonging to the square-integrable space $\mathcal{L}^2$, that means that at $x \to \pm\infty$, then, $\phi \to 0$. So, Fourier transform is define as

$$\mathcal{F}_x(\phi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\kappa x} \phi dx. \tag{A.1}$$

The first identity we are going to need is the Fourier transform of the derivative of a function

$$\mathcal{F}_x\left(\frac{\partial \phi}{\partial x}\right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\kappa x} \frac{\partial \phi}{\partial x} dx, \tag{A.2}$$

integrating by parts and using the fact that the function is square-integrable, we get that

$$\mathcal{F}_x\left(\frac{\partial \phi}{\partial x}\right) = -i\kappa \mathcal{F}_x(\phi). \tag{A.3}$$

Using this last identity is easy to prove that for a $n \in \mathbb{Z}^+$ we have that

$$\mathcal{F}_x\left(\frac{\partial^n \phi}{\partial x^n}\right) = (-i\kappa)^n \mathcal{F}_x(\phi). \tag{A.4}$$

Next, let's study the Fourier transform of the product of the function with its respective variable

$$\mathcal{F}_x(x\phi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\kappa x} x\phi dx, \tag{A.5}$$

then, we rewrite the above expression as

$$\mathcal{F}_x(x\phi) = -\frac{i}{\sqrt{2\pi}}\frac{\partial}{\partial\kappa}\int\limits_{\mathbb{R}} e^{i\kappa x}\phi dx, \tag{A.6}$$

therefore we have that

$$\mathcal{F}_x(x\phi) = -i\frac{\partial}{\partial\kappa}\mathcal{F}_x(\phi), \tag{A.7}$$

and with this result we have for some $n \in \mathbb{Z}^+$ that

$$\mathcal{F}_x(x^n\phi) = \left(-i\frac{\partial}{\partial\kappa}\right)^n \mathcal{F}_x(\phi). \tag{A.8}$$

We continue applying the Fourier transform on the harmonic oscillator function, that is,

$$-\frac{d^2\psi}{dx^2} + x^2\psi = \epsilon_n\psi, \tag{A.9}$$

where $\epsilon_n \in \mathbb{R}$ is a constant define as $\epsilon_n = 2E_n/\hbar\omega_c$ and

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right). \tag{A.10}$$

Applying the Fourier transform to (A.9) and denoting the solution in the Fourier space "κ" as

$$\mathcal{F}_x(\phi) = \overline{\phi}(\kappa), \tag{A.11}$$

we can write

$$-\mathcal{F}_x\left(\frac{d^2\phi}{dx^2}\right) + \mathcal{F}_x\left(x^2\phi\right) = \epsilon_n\mathcal{F}_x(\phi), \tag{A.12}$$

then we use the identities (A.4) and (A.8) both of them with $n = 2$ and rearranging it we write

$$-\frac{\partial^2\overline{\phi}}{\partial\kappa^2} + \kappa^2\overline{\phi} = \epsilon_n\overline{\phi}. \tag{A.13}$$

Hence, the Fourier transform of a harmonic oscillator in the space x is another harmonic oscillator in the space κ. This means that if the solution of (A.9) in the real space is

$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{x^2}{2}\right)H_n(x), \tag{A.14}$$

where H_n are the Hermite polynomials, by definition (A.11) we have that

$$\overline{\phi}_n(\kappa) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} e^{i\kappa x} \phi_n(x) dx, \tag{A.15}$$

that is

$$\frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} e^{i\kappa x} \phi_n(x) dx = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{\kappa^2}{2}\right) H_n(\kappa). \tag{A.16}$$

Then, using the inverse Fourier transform, define as

$$\mathcal{F}_\kappa^{-1}(\phi_n(\kappa)) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} e^{-i\kappa x} \phi_n(\kappa) d\kappa, \tag{A.17}$$

that has the following property

$$\mathcal{F}_\kappa^{-1}\left(\mathcal{F}_x(f(x))\right) = f(x), \tag{A.18}$$

is easy to prove that

$$\phi_n(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} e^{-i\kappa x} \overline{\phi}_n(\kappa) d\kappa, \tag{A.19}$$

meaning that the inverse Fourier transform of a harmonic oscillator in the space κ is another harmonic oscillator in the space x.

A.2 Proof that $\hat{\mathbf{H}} f_n^j = E_n f_n^j$ for Landau's gauge

Here we want to prove that the expression (2.85)

$$f_n^{j+1}(x,y) = m\omega_c\left(\hbar j f_n^{j-1} + (ix-y) f_n^j\right) - i\sqrt{2nm\omega_c\hbar} f_{n-1}^j, \tag{A.20}$$

satisfies the eigenvalue equality $\hat{\mathbf{H}} f_n^{j+1} = E_n f_n^{j+1}$ where the Hamiltonian is of the form (2.52)

$$\hat{\mathbf{H}} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 + 2m\omega_c y\hat{p}_x + m^2\omega_c^2 y^2\right), \tag{A.21}$$

and eigenvalues of

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right). \tag{A.22}$$

First, we must establish that this proof is done by induction, so we have already proved that for $j = 0$ and $j = 1$ the expression

$$\hat{\mathbf{H}} f_n^j = E_n f_n^j, \tag{A.23}$$

holds. Second, its useful to write down a couple of expressions that will be helpful for this task. The reader can verify that the following equalities holds

$$\hat{\mathbf{H}}(x f_n^j) = -i\frac{\hbar}{m} f_n^{j+1} - i\hbar\omega_c y f_n^j + x E_n f_n^j, \tag{A.24}$$

and

$$\hat{\mathbf{H}}(y f_n^j) = -i\frac{\hbar}{m}\hat{p}_y f_n^j + y E_n f_n^j. \tag{A.25}$$

Applying the Hamiltonian (2.52) we have that

$$\hat{\mathbf{H}} f_n^{j+1} = m\omega_c\left(\hbar j\hat{\mathbf{H}} f_n^{j-1} + (i\hat{\mathbf{H}}(x f_n^j) - \hat{\mathbf{H}}(y f_n^j))\right) - i\sqrt{2nm\omega_c\hbar}\hat{\mathbf{H}} f_{n-1}^j, \tag{A.26}$$

using (A.23), substituting expressions (A.24), (A.25),

$$\hat{\mathbf{H}} f_n^{j+1} = m\omega_c\left(\hbar j E_n f_n^{j-1} + (ix-y) E_n f_n^j\right) + \hbar\omega_c\left(f_n^{j+1} + m\omega_c y f_n^j + i\hat{p}_y f_n^j\right) - i\sqrt{2nm\omega_c\hbar} E_{n-1} f_{n-1}^j, \tag{A.27}$$

analyzing the following sum of terms by substituting the expression (A.20) we can see that

$$\hbar\omega_c f_n^{j+1} - i\sqrt{2nm\omega_c\hbar}E_{n-1}f_{n-1}^j = \hbar m\omega_c^2\left(\hbar j f_n^{j-1} + (ix-y)f_n^j\right) - i\sqrt{2nm\omega_c\hbar}(E_{n-1}+\hbar\omega_c)f_{n-1}^j \quad \text{(A.28)}$$

then is easy to see that $E_{n-1} + \hbar\omega_c = E_n$ and simplifying we have

$$\hat{\mathbf{H}}f_n^{j+1} = E_n\left(m\omega_c\left(\hbar j f_n^{j-1} + (ix-y)f_n^j\right) - i\sqrt{2nm\omega_c\hbar}f_{n-1}^j\right) + \hbar\omega_c\left(m\omega_c\hbar j f_n^{j-1} + m\omega_c ix f_n^j + i\hat{p}_y f_n^j\right), \quad \text{(A.29)}$$

then we can identify the above expression as

$$\hat{\mathbf{H}}f_n^{j+1} = E_n f_n^{j+1} + \hbar\omega_c\left(m\omega_c\hbar j f_n^{j-1} + m\omega_c ix f_n^j + i\hat{p}_y f_n^j\right). \quad \text{(A.30)}$$

Finally, note that as (2.79) holds as true, then, necessarily

$$m\omega_c\hbar j f_n^{j-1} + m\omega_c ix f_n^j + i\hat{p}_y f_n^j = 0. \quad \text{(A.31)}$$

Note that this prove is valid for $j = 0, 1$ which implies that is valid too for $j = 2$

$$\hat{\mathbf{H}}f_n^2 = E_n f_n^2, \quad \text{(A.32)}$$

then the prove holds for the next values of j and for induction this is valid for any $j \in \mathbb{Z}^+$.

Appendix B

Calculation details for the symmetric gauge

B.1 Normalization constant for the symmetric gauge

Being the functions

$$\varphi_n(x,y) = A\exp\left(-\alpha(x^2+y^2) - \lambda(x+iy)\right)\Big(2\alpha(x-iy)+\lambda\Big)^n, \tag{B.1}$$

we want to determine the normalization constant A such that

$$\langle\varphi_n|\varphi_n\rangle = 1. \tag{B.2}$$

So, the inner product can be written as

$$\langle\varphi_n|\varphi_n\rangle = A_n^2 \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \exp\left(-2\alpha(x^2+y^2) - \lambda^*(x-iy) - \lambda(x+iy)\right)(2\alpha(x+iy)+\lambda^*)^n(2\alpha(x-iy)+\lambda)^n dxdy, \tag{B.3}$$

then, $\lambda = Re(\lambda) + iIm(\lambda)$ and the next equalities can be written

$$-2\alpha(x^2+y^2) - \lambda^*(x-iy) - \lambda(x+iy) = -2\alpha\left(\left(x+\frac{Re(\lambda)}{2\alpha}\right)^2 + \left(y-\frac{Im(\lambda)}{2\alpha}\right)^2\right) + \frac{\lambda^*\lambda}{2\alpha}, \tag{B.4}$$

and also

$$(2\alpha(x-iy)+\lambda^*)(2\alpha(x+iy)+\lambda) = (2\alpha)^2\left(\left(x+\frac{Re(\lambda)}{2\alpha}\right)^2 + \left(y-\frac{Im(\lambda)}{2\alpha}\right)^2\right), \tag{B.5}$$

Then we can rewrite the integral (B.3) as follows

$$\langle\varphi_n|\varphi_n\rangle = A^2\exp\left(\frac{\lambda^*\lambda}{2\alpha}\right)(2\alpha)^{2n}\times$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-2\alpha\left(\left(x+\frac{Re(\lambda)}{2\alpha}\right)^2+\left(y-\frac{Im(\lambda)}{2\alpha}\right)^2\right)\right)\left(\left(x+\frac{Re(\lambda)}{2\alpha}\right)^2+\left(y-\frac{Im(\lambda)}{2\alpha}\right)^2\right)^n dxdy, \tag{B.6}$$

here, we can use the binomial expression

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k \tag{B.7}$$

where the coefficients are define as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \tag{B.8}$$

we use it to write down

$$\left(\left(x+\frac{Re(\lambda)}{2\alpha}\right)^2+\left(y-\frac{Im(\lambda)}{2\alpha}\right)^2\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(x+\frac{Re(\lambda)}{2\alpha}\right)^{2(n-k)}\left(y-\frac{Im(\lambda)}{2\alpha}\right)^{2k}, \tag{B.9}$$

then, the integral to be solved takes the next form

$$\langle\varphi_n|\varphi_n\rangle = A^2\exp\left(\frac{\lambda^*\lambda}{2\alpha}\right)(2\alpha)^{2n}\sum_{k=0}^{n}\binom{n}{k}\int_{-\infty}^{\infty}\exp\left(-2\alpha\left(x+\frac{Re(\lambda)}{2\alpha}\right)^2\right)\left(x+\frac{Re(\lambda)}{2\alpha}\right)^{2(n-k)}dx\times$$

$$\times\int_{-\infty}^{\infty}\exp\left(-2\alpha\left(y+\frac{Im(\lambda)}{2\alpha}\right)^2\right)\left(y-\frac{Im(\lambda)}{2\alpha}\right)^{2k}dy. \tag{B.10}$$

Now, we focus on solving the integral respect of x making the change of variable

$$\xi = \sqrt{2\alpha}\left(x+\frac{Re(\lambda)}{2\alpha}\right), \tag{B.11}$$

we write

$$\int_{-\infty}^{\infty} \exp\left(-2\alpha\left(x+\frac{Re(\lambda)}{2\alpha}\right)^2\right)\left(x+\frac{Re(\lambda)}{2\alpha}\right)^{2(n-k)} dx = \frac{1}{(2\alpha)^{n-k+1/2}}\int_{-\infty}^{\infty} e^{-\xi^2}\xi^{2(n-k)}d\xi \tag{B.12}$$

then, we use the definition of the *Gamma function*

$$\int_{-\infty}^{\infty} e^{-\xi^2}\xi^m d\xi = \frac{1}{2}((-1)^m+1)\Gamma\left(\frac{m+1}{2}\right), \quad \text{for} \quad Re(m) > -1, \tag{B.13}$$

with this we can write

$$\int_{-\infty}^{\infty} \exp\left(-2\alpha\left(x+\frac{Re(\lambda)}{2\alpha}\right)^2\right)\left(x+\frac{Re(\lambda)}{2\alpha}\right)^{2(n-k)} dx = \frac{1}{(2\alpha)^{n-k+1/2}}\Gamma\left(n-k+\frac{1}{2}\right). \tag{B.14}$$

Similarly, we can solve the integral respect the variable y to obtain

$$\int_{-\infty}^{\infty} \exp\left(-2\alpha\left(y+\frac{Im(\lambda)}{2\alpha}\right)^2\right)\left(y-\frac{Im(\lambda)}{2\alpha}\right)^{2k} dy = \frac{1}{(2\alpha)^{k+1/2}}\Gamma\left(k+\frac{1}{2}\right). \tag{B.15}$$

Therefore, substituting this last two results in the inner product expression (B.10) and simplyfing we have that

$$\langle\varphi_n|\varphi_n\rangle = A^2\exp\left(\frac{\lambda^*\lambda}{2\alpha}\right)(2\alpha)^{n-1}\sum_{k=0}^{n}\binom{n}{k}\Gamma\left(n-k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right), \tag{B.16}$$

then, due to the normalization condition we have that

$$A^2\exp\left(\frac{\lambda^*\lambda}{2\alpha}\right)(2\alpha)^{n-1}\sum_{k=0}^{n}\binom{n}{k}\Gamma\left(n-k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right) = 1. \tag{B.17}$$

Continuing simplifying, it can prove that

$$\sum_{k=0}^{n}\binom{n}{k}\Gamma\left(n-k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right) = \pi n!. \tag{B.18}$$

and with this, the normalization constant has the form of

$$A_n = \frac{e^{-|\lambda|^2/4\alpha}}{\sqrt{(2\alpha)^{n-1}\pi n!}}. \tag{B.19}$$

B.2 Degenerated eigenfunctions of symmetric gauge

Here, we show the calculation details of the degenerated eigenfunctions for the symmetric gauge. Being the solution to the two dimensional Hamiltonian (2.129)

$$\varphi_n(x,y) = A_n \exp\left(-\alpha(x^2+y^2) - \lambda(x+iy)\right)\Big(2\alpha(x-iy)+\lambda\Big)^n, \tag{B.20}$$

where A_n is define as (2.160). Then we have the following eigenfunction generators

$$\hat{\pi}'_x = \hat{p}_x - \frac{m\omega_c}{2}y, \tag{B.21}$$

$$\hat{\pi}'_y = \hat{p}_y + \frac{m\omega_c}{2}x, \tag{B.22}$$

and

$$\hat{L}_z = x\hat{p}_y - y\hat{p}_x. \tag{B.23}$$

We begin calculating the eigenfunctions generated by applying the operator (B.21).We are going to denote this eigenfunctions as follows

$$(\hat{\pi}'_x)^j \varphi_n(x,y) = f_n^j(x,y), \quad j = 0,1,2... \tag{B.24}$$

where j denote the j-th application of the operator and the next definition was made $f_n^0(x,y) = \varphi_n(x,y)$. The first application of this operator will give the eigenfunction

$$f_n^1(x,y) = \hat{\pi}'_x f_n^0(x,y) = \left(\frac{m\omega_c}{2}(ix-y) + i\hbar\lambda\right) f_n^0 - i\sqrt{\frac{m\omega_c\hbar n}{2}} f_{n-1}^0. \tag{B.25}$$

Now, we use this expression to calculate the function for the $j+1$ application of the operator $\hat{\pi}'_x$. First, note that if two operators, $\hat{A}$ and $\hat{B}$, commute with each other, that is, $[\hat{A},\hat{B}] = 0$, then, the next property holds

$$(\hat{A}+\hat{B})^j = \sum_{m=0}^{j} \binom{j}{m} \hat{A}^{j-m}\hat{B}^m, \tag{B.26}$$

where the binomial coefficient (B.8) is used. Defining the constant $a = m\omega_c/2$, we can write down

$$(\hat{\pi}'_x)^j f_n^1 = (\hat{p}_x - ay)^j f_n^1 = \sum_{m=0}^{j} \binom{j}{m} (\hat{p}_x)^{j-m} f_n^1 (-ay)^m, \tag{B.27}$$

second, is helpful to calculate the expression

$$(\hat{p}_x)^{\mathcal{M}} f_n^1 = \frac{m\omega_c}{2}(-i\hbar)^{\mathcal{M}} \frac{\partial^{\mathcal{M}}}{\partial x^{\mathcal{M}}}\Big((ix-y)f_n^0\Big) + i\hbar\lambda \hat{p}_x^{\mathcal{M}} f_n^0 - i\sqrt{\frac{m\omega_c\hbar n}{2}} \hat{p}_x^{\mathcal{M}} f_{n-1}^0, \tag{B.28}$$

then, using the expression for the derivative of the product of two functions, eq. (2.82), we can write

$$(\hat{p}_x)^{\mathcal{M}} f_n^1 = \frac{m\omega_c}{2}\left((ix-y)\hat{p}_x^{\mathcal{M}} f_n^0 + \hbar\mathcal{M}\hat{p}_x^{\mathcal{M}-1} f_n^0\right) + i\hbar\lambda\hat{p}_x^{\mathcal{M}} f_n^0 - i\sqrt{\frac{m\omega_c\hbar n}{2}}\hat{p}_x^{\mathcal{M}} f_{n-1}^0, \tag{B.29}$$

substituting this result in eq. (B.27) with $\mathcal{M} = j - m$, we have that

$$(\hat{\pi}'_x)^j f_n^1 = \sum_{m=0}^{j}\binom{j}{m}\left(a\left((ix-y)\hat{p}_x^{j-m} f_n^0 + \hbar(j-m)\hat{p}_x^{j-m-1} f_n^0\right) + i\hbar\lambda\hat{p}_x^{j-m} f_n^0 - i\sqrt{a\hbar n}\hat{p}_x^{j-m} f_{n-1}^0\right)(-ay)^m, \tag{B.30}$$

now, is possible to do identifications with the expression (B.27) to notice that the above equality can be rewritten as

$$(\hat{\pi}'_x)^j f_n^1 = a\left((ix-y)(\hat{\pi}'_x)^j f_n^0 + \hbar\sum_{m=0}^{j}\binom{j}{m}(j-m)\hat{p}_x^{j-m-1} f_n^0(-ay)^m\right) + i\hbar\lambda(\hat{\pi}'_x)^j f_n^0 - i\sqrt{a\hbar n}(\hat{\pi}'_x)^j f_{n-1}^0. \tag{B.31}$$

Following up this calculation, we realize that the summation left has not contribution when $m = j$, so it can be rewritten as

$$\sum_{m=0}^{j}\binom{j}{m}(j-m)\hat{p}_x^{j-m-1} f_n^0(-ay)^m = \sum_{m=0}^{j-1}\binom{j}{m}(j-m)\hat{p}_x^{j-m-1} f_n^0(-ay)^m, \tag{B.32}$$

then, we note that

$$\binom{j}{m}(j-m) = j\binom{j-1}{m}, \tag{B.33}$$

this allow us to write

$$j\sum_{m=0}^{j-1}\binom{j-1}{m}\hat{p}_x^{j-m-1} f_n^0(-ay)^m = j(\hat{\pi}'_x)^{j-1} f_n^0, \tag{B.34}$$

finally, substituting the above expression and using the definition (B.24)

$$f_n^{j+1} = a\left((ix-y)f_n^j + j\hbar f_n^{j-1}\right) + i\hbar\lambda f_n^j - i\sqrt{a\hbar n} f_{n-1}^j. \tag{B.35}$$

Is not difficult to figure out that the deduction for the functions obtained applying the operator (B.22) is similar to the one we have done. Hence, we are going to write down the results only and the details are left for the interested reader. Lets denote this new functions as

$$(\hat{\pi}'_y)^j \varphi_n(x, y) = g_n^j(x, y), \quad j = 0, 1, 2... \tag{B.36}$$

where, again, j denote the j-th application of the operator and the next definition was made $g_n^0(x, y) = \varphi_n(x, y)$. The first application of this operator will give the eigenfunction

$$g_n^1(x, y) = \hat{\pi}'_y g_n^0(x, y) = \left(\frac{m\omega_c}{2}(x + iy) - \hbar\lambda\right) g_n^0 - \sqrt{\frac{m\omega_c\hbar n}{2}} g_{n-1}^0, \tag{B.37}$$

which can be used to calculate the expression

$$g_n^{j+1} = a\Big((x + iy)g_n^j + j\hbar g_n^{j-1}\Big) - \hbar\lambda g_n^j - \sqrt{a\hbar n} g_{n-1}^j. \tag{B.38}$$

Finally, lets analyze the eigenfunctions obtained with the angular momentum defined in the eq. (B.23)

$$\hat{L}_z = x\hat{p}_y - y\hat{p}_x. \tag{B.39}$$

In this case, we adopt the following notation

$$(\hat{L}_z)^j \varphi_n(x, y) = \mathcal{L}_n^j(x, y), \quad j = 0, 1, 2... \tag{B.40}$$

the first degenerated function which is obtain by applying this operator can be written as

$$\mathcal{L}_n^1 = -\hbar\lambda(x + iy)\varphi_n - \sqrt{\frac{m\omega_c\hbar n}{2}}(x - iy)\varphi_{n-1}, \tag{B.41}$$

this expression can be rewritten in polar coordinates, using $z = re^{i\theta}$ with $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$ and $\varphi = \varphi(r, \theta)$, then

$$\mathcal{L}_n^1 = -\hbar\lambda re^{i\theta}\varphi_n - \sqrt{\frac{m\omega_c\hbar n}{2}} re^{-i\theta}\varphi_{n-1}. \tag{B.42}$$

The later expression is useful for our purpose of finding a general expression of the consecutive angular momentum applications, since in poolar coordinates this operator can has the following form

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\theta}, \tag{B.43}$$

therefore, the $j + 1$'th application of this operator can be done as follows

$$\mathcal{L}_n^{j+1} = (-i\hbar)^j \left(-\hbar\lambda r\frac{\partial^j}{\partial\theta^j}\left(e^{i\theta}\varphi_n\right) - \sqrt{\frac{m\omega_c\hbar n}{2}} r\frac{\partial^j}{\partial\theta^j}\left(e^{-i\theta}\varphi_{n-1}\right)\right) \tag{B.44}$$

then, using the formula for the derivative of two functinos eq.(2.82) we have

$$\frac{\partial^j}{\partial\theta^j}(e^{i\theta}\varphi_n) = e^{i\theta}\sum_{m=0}^{j}\binom{j}{m}\frac{(i)^{j-m}}{(-i\hbar)^m}\hat{L}_z^m\varphi_n, \tag{B.45}$$

similarly

$$\frac{\partial^j}{\partial\theta^j}(e^{-i\theta}\varphi_{n-1}) = e^{-i\theta}\sum_{m=0}^{j}\binom{j}{m}\frac{(-i)^{j-m}}{(-i\hbar)^m}\hat{L}_z^m\varphi_{n-1}, \tag{B.46}$$

substituting them and simplifying

$$\mathcal{L}_n^{j+1} = \sum_{m=0}^{j}\binom{j}{m}\left(-\lambda r e^{i\theta}(\hbar)^{j-m+1}\hat{L}_z^m\varphi_n - \sqrt{\frac{m\omega_c\hbar n}{2}}re^{-i\theta}(-\hbar)^{j-m}\hat{L}_z^m\varphi_{n-1}\right) \tag{B.47}$$

defining the constants

$$c_m^j = \binom{j}{m}(\hbar)^{j-m}, \tag{B.48}$$

and

$$d_m^j = \binom{j}{m}(-\hbar)^{j-m}, \tag{B.49}$$

then, using the definition (B.40), the general expression for the eigenfunctions given by the angular momentum can be written in cartesian coordinates as

$$\mathcal{L}_n^{j+1} = \sum_{m=0}^{j}\left(-\hbar\lambda(x+iy)c_m^j\mathcal{L}_n^m - \sqrt{\frac{m\omega_c\hbar n}{2}}(x-iy)d_m^j\mathcal{L}_{n-1}^m\right). \tag{B.50}$$

B.3 Proof that the degeneration expressions satisfies the eigenvalue equation

So far, we have only deduced the expression for the degenerated eigenfunctions using the conserved operators, however, it is a healty practice to prove that they satisfy the eigenvalue equation $H\psi = E\psi$ with the two dimenssional Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 - m\omega_c\hat{L}_z + \frac{m^2\omega_c^2}{4}(x^2+y^2)\right), \tag{B.51}$$

having the eigenvalues

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right). \tag{B.52}$$

We beging with the functions (B.35)

$$f_n^{j+1} = a\left((ix-y)f_n^j + j\hbar f_n^{j-1}\right) + i\hbar\lambda f_n^j - i\sqrt{a\hbar n}f_{n-1}^j. \tag{B.53}$$

Like the case for the degeneration of Landau's gauge, presented in the appendix (A.2), we are going to do this proof by induction. Assume that we have two eigenfunctions, such that

$$\hat{\mathcal{H}}f_n^{j-1} = E_n f_n^{j-1}, \quad \hat{\mathcal{H}}f_n^j = E_n f_n^j \tag{B.54}$$

then, applying the Hamiltonian (B.51)

$$\hat{\mathcal{H}}f_n^{j+1} = a\left(\hat{\mathcal{H}}\left[(ix-y)f_n^j\right] + j\hbar\hat{\mathcal{H}}f_n^{j-1}\right) + i\hbar\lambda\hat{\mathcal{H}}f_n^j - i\sqrt{a\hbar n}\hat{\mathcal{H}}f_{n-1}^j, \tag{B.55}$$

then, is necessary to calculate the expression

$$\hat{\mathcal{H}}\left[(ix-y)f_n^j\right] = \frac{1}{2m}\left(\hat{p}_x^2\left[(ix-y)f_n^j\right] + \hat{p}_y^2\left[(ix-y)f_n^j\right] - m\omega_c\hat{L}_z\left[(ix-y)f_n^j\right] + \frac{m^2\omega_c^2}{4}(x^2+y^2)\left[(ix-y)f_n^j\right]\right). \tag{B.56}$$

Is useful to calculate the following expression,

$$\hat{p}_x\left[(ix-y)f_n^j\right] = \hbar f_n^j + (ix-y)\hat{p}_x f_n^j, \tag{B.57}$$

$$\hat{p}_x^2\left[(ix-y)f_n^j\right] = 2\hbar\hat{p}_x f_n^j + (ix-y)\hat{p}_x^2 f_n^j, \tag{B.58}$$

$$\hat{p}_y\left[(ix-y)f_n^j\right] = i\hbar f_n^j + (ix-y)\hat{p}_y f_n^j, \tag{B.59}$$

$$\hat{p}_y^2\left[(ix-y)f_n^j\right] = 2i\hbar\hat{p}_y f_n^j + (ix-y)\hat{p}_y^2 f_n^j, \tag{B.60}$$

and

$$\hat{L}_z\Big[(ix-y)f_n^j\Big] = i\hbar x f_n^j - \hbar y f_n^j + (ix-y)\hat{L}_z f_n^j. \tag{B.61}$$

substituting all the above equalities in the equality (B.56)

$$\hat{\mathcal{H}}\Big[(ix-y)f_n^j\Big] = \frac{\hbar}{m}\left(\hat{\pi}_x f_n^j + i\hat{\pi}_y f_n^j\right) + E_n(ix-y)f_n^j. \tag{B.62}$$

where the definition of the operators (2.164) was used,

$$\hat{\pi}_x = \hat{p}_x + \frac{m\omega_c}{2}y, \qquad \hat{\pi}_y = \hat{p}_y - \frac{m\omega_c}{2}x. \tag{B.63}$$

Hence, we figure out that

$$\hat{\mathcal{H}}f_n^{j+1} = \hat{\mathcal{H}}f_n^{j+1} + \frac{a\hbar}{m}\left(\hat{\pi}_x f_n^j + i\hat{\pi}_y f_n^j\right) + i\hbar\omega_c\sqrt{a\hbar n}f_{n-1}^j. \tag{B.64}$$

Then, using the commutation relations (2.168), (2.169) and the definition (B.24), we realized that the next equality can be written

$$\left(\hat{\pi}_x f_n^j + i\hat{\pi}_y f_n^j\right) = (\hat{\pi}_x')^j\left(\hat{\pi}_x\varphi_n + i\hat{\pi}_y\varphi_n\right) = -2i\hbar\sqrt{2\alpha n}f_{n-1}^j, \tag{B.65}$$

Finally, realizing that $a = 2\alpha\hbar$ we have that

$$\frac{a\hbar}{m}\left(\hat{\pi}_x f_n^j + i\hat{\pi}_y f_n^j\right) + i\hbar\omega_c\sqrt{a\hbar n}f_{n-1}^j = -i\hbar\omega_c\sqrt{a\hbar n}f_{n-1}^j + i\hbar\omega_c\sqrt{a\hbar n}f_{n-1}^j = 0, \tag{B.66}$$

therefore, it satisfies the eigenvalue equation

$$\hat{\mathcal{H}}f_n^{j+1} = E_n f_n^{j+1}. \tag{B.67}$$

The proof for the expression (B.38) defining the functions $g_n^{j+1}(x,y)$ is similar to the one we have just done here, therefore, is left for the interested reader.
Finally, we are going to calculate the action of the Hamiltonian (B.51) over the functions (B.50). To accomplish this task, is helpful to write down the following equalities

$$\hat{p}_x\mathcal{L}_n^{j+1} = \sum_{m=0}^{j}\Big(i\hbar^2\lambda c_m^j\mathcal{L}_n^m + i\hbar\sqrt{a\hbar n}d_m^j\mathcal{L}_{n-1}^m - \hbar\lambda(x+iy)c_m^j\hat{p}_x\mathcal{L}_n^m - \sqrt{a\hbar n}(x-iy)d_m^j\hat{p}_x\mathcal{L}_{n-1}^m\Big), \tag{B.68}$$

$$\hat{p}_x^2\mathcal{L}_n^{j+1} = \sum_{m=0}^{j}\left(2i\hbar^2\lambda c_m^j\hat{p}_x\mathcal{L}_n^m + 2i\hbar\sqrt{a\hbar n}d_m^j\hat{p}_x\mathcal{L}_{n-1}^m - \hbar\lambda(x+iy)c_m^j\hat{p}_x^2\mathcal{L}_n^m - \sqrt{a\hbar n}(x-iy)d_m^j\hat{p}_x^2\mathcal{L}_{n-1}^m\right), \tag{B.69}$$

$$\hat{p}_y\mathcal{L}_n^{j+1} = \sum_{m=0}^{j}\left(-\hbar^2\lambda c_m^j\mathcal{L}_n^m + \hbar\sqrt{a\hbar n}d_m^j\mathcal{L}_{n-1}^m - \hbar\lambda(x+iy)c_m^j\hat{p}_y\mathcal{L}_n^m - \sqrt{a\hbar n}(x-iy)d_m^j\hat{p}_y\mathcal{L}_{n-1}^m\right), \tag{B.70}$$

and

$$\hat{p}_y^2\mathcal{L}_n^{j+1} = \sum_{m=0}^{j}\left(-2\hbar^2\lambda c_m^j\hat{p}_y\mathcal{L}_n^m + 2\hbar\sqrt{a\hbar n}d_m^j\hat{p}_y\mathcal{L}_{n-1}^m - \hbar\lambda(x+iy)c_m^j\hat{p}_y^2\mathcal{L}_n^m - \sqrt{a\hbar n}(x-iy)d_m^j\hat{p}_y^2\mathcal{L}_{n-1}^m\right), \tag{B.71}$$

substituting all the above expressions in the Hamiltonian (B.51) doing some rearrangements and identifications, one could get the following expression

$$\hat{\mathcal{H}}f_n^{j+1} = E_nf_n^{j+1} + \frac{\hbar i}{m}\sum_{m=0}^{j}\left(\hbar\lambda c_m^j(\hat{\pi}_x + i\hat{\pi}_y)\mathcal{L}_n^m + \sqrt{a\hbar n}d_m^j(\hat{\pi}_x - i\hat{\pi}_y)\mathcal{L}_{n-1}^m - i2a\sqrt{a\hbar n}d_m^j(x-iy)\mathcal{L}_{n-1}^m\right), \tag{B.72}$$

where the definition of the operators (2.164) was used

$$\hat{\pi}_x = \hat{p}_x + \frac{m\omega_c}{2}y, \qquad \hat{\pi}_y = \hat{p}_y - \frac{m\omega_c}{2}x, \tag{B.73}$$

where, due to the commutation, is deduced that

$$\sum_{m=0}^{j}\left(\hbar\lambda c_m^j(\hat{\pi}_x + i\hat{\pi}_y)\mathcal{L}_n^m + \sqrt{a\hbar n}d_m^j(\hat{\pi}_x - i\hat{\pi}_y)\mathcal{L}_{n-1}^m - i2a\sqrt{a\hbar n}d_m^j(x-iy)\mathcal{L}_{n-1}^m\right) = 0. \tag{B.74}$$

B.4 Commutation relations for the momentum operators

Here we are going to deduce a couple of commutation relations that are useful to find a second eigenvalue expression regarding the index of degeneration j for the solutions found using the Landau's gauge and symmetric gauge. First, lets recall that for two operators that commute with each other, $[\hat{A}, \hat{B}] = 0$ is possible to use the binomial expansion (B.26)

$$(\hat{A}+\hat{B})^j = \sum_{m=0}^{j} \binom{j}{m} \hat{A}^{j-m}\hat{B}^m, \tag{B.75}$$

using this, we can write down

$$\left(\hat{p}_x - \frac{m\omega_c}{2}y\right)^{j+1} \hat{p}_y = \sum_{M=0}^{j+1} \binom{j+1}{M} \hat{p}_x^{j+1-M} \left(-\frac{m\omega_c}{2}y\right)^M \hat{p}_y, \tag{B.76}$$

then, is easy to calculate the commutator

$$[y^M, \hat{p}_y] = i\hbar M y^{M-1}, \quad M \in \mathbb{Z}^+, \tag{B.77}$$

with its aid we can write

$$\left(\hat{p}_x - \frac{m\omega_c}{2}y\right)^{j+1} \hat{p}_y = \sum_{M=0}^{j+1} \binom{j+1}{M} \hat{p}_x^{j+1-M} \left(-\frac{m\omega_c}{2}\right)^M (i\hbar M y^{M-1} + \hat{p}_y y^M), \tag{B.78}$$

so, we can rewrite it as

$$\left(\hat{p}_x - \frac{m\omega_c}{2}y\right)^{j+1} \hat{p}_y = i\hbar \sum_{M=0}^{j+1} \binom{j+1}{M} \hat{p}_x^{j+1-M} \left(-\frac{m\omega_c}{2}\right)^M M y^{M-1} + \hat{p}_y \left(\hat{p}_x - \frac{m\omega_c}{2}y\right)^{j+1}. \tag{B.79}$$

Now, analyzing the first term on the right hand of the above equality we realize that the term when $M=0$ has no contribution on the sumation, thus,

$$i\hbar \sum_{M=0}^{j+1} \binom{j+1}{M} \hat{p}_x^{j+1-M} \left(-\frac{m\omega_c}{2}\right)^M M y^{M-1} = i\hbar \sum_{M=1}^{j+1} \binom{j+1}{M} \hat{p}_x^{j+1-M} \left(-\frac{m\omega_c}{2}\right)^M M y^{M-1}, \tag{B.80}$$

then, changing the index for $Q = M-1$

$$i\hbar \sum_{Q=0}^{j} \binom{j+1}{Q+1} \hat{p}_x^{j-Q} \left(-\frac{m\omega_c}{2}\right)^{Q+1} (Q+1) y^Q = i\hbar (j+1) \left(-\frac{m\omega_c}{2}\right) \sum_{Q=0}^{j} \binom{j}{Q} \hat{p}_x^{j-Q} \left(-\frac{m\omega_c}{2}y\right)^Q, \tag{B.81}$$

where in the last equality the following property of the binomial coefficient was used

$$\binom{j+1}{Q+1} = \frac{(j+1)!}{(Q+1)!(j-Q)!} = \frac{j+1}{Q+1}\frac{j!}{Q!(j-Q)!} = \frac{j+1}{Q+1}\binom{j}{Q}. \tag{B.82}$$

Hence, we can write

$$(\hat{\pi}'_x)^{j+1}\hat{p}_y = -i\hbar(j+1)\left(\frac{m\omega_c}{2}\right)(\hat{\pi}'_x)^j + \hat{p}_y(\hat{\pi}'_x)^{j+1}, \tag{B.83}$$

or, writing it as a commutator

$$[\hat{p}_y, (\hat{\pi}'_x)^{j+1}] = i\hbar(j+1)\left(\frac{m\omega_c}{2}\right)(\hat{\pi}'_x)^j. \tag{B.84}$$

We continue the analysis calculating the next expression

$$(\hat{\pi}'_x)^{j+1}x = \left(\hat{p}_x - \frac{m\omega_c}{2}y\right)^{j+1} x = \sum_{M=0}^{j+1}\binom{j+1}{M}\hat{p}_x^{j+1-M}\left(-\frac{m\omega_c}{2}y\right)^M x, \tag{B.85}$$

here, we use the following commutation relation

$$[x, \hat{p}_x^M] = i\hbar M\hat{p}_x^{M-1}, \quad M \in \mathbb{Z}^+, \tag{B.86}$$

we can write

$$\sum_{M=0}^{j+1}\binom{j+1}{M}\hat{p}_x^{j+1-M}\left(-\frac{m\omega_c}{2}y\right)^M x = \sum_{M=0}^{j+1}\binom{j+1}{M}(x\hat{p}_x^{j+1-M} - i\hbar(j+1-M)\hat{p}_x^{j-M})\left(-\frac{m\omega_c}{2}y\right)^M \tag{B.87}$$

$$= x\left(\hat{p}_x - \frac{m\omega_c}{2}y\right)^{j+1} - i\hbar\sum_{M=0}^{j+1}\binom{j+1}{M}(j+1-M)\hat{p}_x^{j-M}\left(-\frac{m\omega_c}{2}y\right)^M \tag{B.88}$$

noting that when $M = j+1$ in the second term on the right hand side of the above equality has no contribution to the sum, we have that

$$i\hbar\sum_{M=0}^{j}\binom{j+1}{M}(j+1-M)\hat{p}_x^{j-M}\left(-\frac{m\omega_c}{2}y\right)^M = i\hbar(j+1)\sum_{M=0}^{j}\binom{j}{M}\hat{p}_x^{j-M}\left(-\frac{m\omega_c}{2}y\right)^M \tag{B.89}$$

where the following property was used

$$\binom{j+1}{M} = \frac{(j+1)!}{M!(j+1-M)!} = \frac{j+1}{(j+1-M)}\frac{j!}{M!(j-M)!} = \frac{j+1}{j+1-M}\binom{j}{M}. \tag{B.90}$$

Hence, we found that

$$(\hat{\pi}'_x)^{j+1}x = x(\hat{\pi}'_x)^{j+1} - i\hbar(j+1)(\hat{\pi}'_x)^j, \tag{B.91}$$

or, rewriting it as a commutation relation

$$[x, (\hat{\pi}'_x)^{j+1}] = i\hbar(j+1)(\hat{\pi}'_x)^j. \tag{B.92}$$

Using the results $(B.84)$ y $(B.92)$ we can write the commutator as follows

$$[\hat{\pi}'_y, (\hat{\pi}'_x)^{j+1}] = i\hbar m\omega_c(j+1)(\hat{\pi}'_x)^j. \tag{B.93}$$

Similarly, the following commutation relations can be obtaine

$$[(\hat{\pi}'_y)^{j+1}, \hat{p}_x] = i\hbar\left(\frac{m\omega_c}{2}\right)(j+1)(\hat{\pi}'_y)^j, \tag{B.94}$$

and

$$[(\hat{\pi}'_y)^{j+1}, y] = -i\hbar(j+1)(\hat{\pi}'_y)^j, \tag{B.95}$$

which can be used to obtain the commutation

$$[(\hat{\pi}'_y)^{j+1}, \hat{\pi}'_x] = i\hbar m\omega_c(j+1)(\hat{\pi}'_y)^j. \tag{B.96}$$

B.5 Calculation of expectation value of the position and position squared

We continue performing the calculation of expectation values that are going to be useful to calculate the density of states. First, we are going to calculate the expectation value of the position coordinate x and we are going to constrain this calculation to the eigenfunctions given by eq.(2.159)

$$\varphi_n(x,y) = \frac{e^{-|\lambda|^2/4\alpha}}{\sqrt{(2\alpha)^{n-1}\pi n!}}\exp\left(-\alpha(x^2+y^2)-\lambda(x+iy)\right)\Big(2\alpha(x-iy)+\lambda\Big)^n, \tag{B.97}$$

therefore, we can write

$$\langle\varphi_n|\, x\, |\varphi_n\rangle = \iint\limits_{\mathbb{R}^2} \varphi_n^* x \varphi_n \, dx\, dy, \tag{B.98}$$

with a similar algebraic process than the one done in section (B.1) the above integral can be written as

$$\langle\varphi_n|\, x\, |\varphi_n\rangle = \frac{(2\alpha)^{n+1}}{\pi n!}\sum_{k=0}^{n}\binom{n}{k}\times$$

$$\times\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} x\exp\left(-2\alpha\left(\left(x+\frac{Re(\lambda)}{2\alpha}\right)^2+\left(y-\frac{Im(\lambda)}{2\alpha}\right)^2\right)\right)\left(x+\frac{Re(\lambda)}{2\alpha}\right)^{2(n-k)}\left(y-\frac{Im(\lambda)}{2\alpha}\right)^{2k} dxdy, \tag{B.99}$$

now, we perform the change of variable

$$\xi_1 = \sqrt{2\alpha}\left(x + \frac{Re(\lambda)}{2\alpha}\right), \tag{B.100}$$

and

$$\xi_2 = \sqrt{2\alpha}\left(y - \frac{Im(\lambda)}{2\alpha}\right), \tag{B.101}$$

then, the integration has the following form

$$\langle\varphi_n| x |\varphi_n\rangle = \frac{1}{\pi n!}\sum_{k=0}^{n}\binom{n}{k}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\left(\frac{\xi_1}{\sqrt{2\alpha}} - \frac{Re(\lambda)}{2\alpha}\right)\exp\left(-\xi_1^2 - \xi_2^2\right)\xi_1^{2(n-k)}\xi_2^{2k}\, d\xi_1\, d\xi_2. \tag{B.102}$$

From this last expression and using the formula eq.(B.13) it can be seen that the following integral vanish

$$\int\limits_{-\infty}^{\infty}\exp\left(-\xi_1^2\right)\xi_1^{2(n-k)+1}\, d\xi_1 = 0, \tag{B.103}$$

because is the integral of an odd function performed over a simmetric interval. Hence, using, again, the expression eq.(B.13) we have the following result

$$\langle\varphi_n| x |\varphi_n\rangle = -\frac{Re(\lambda)}{2\alpha}\frac{1}{\pi n!}\sum_{k=0}^{n}\binom{n}{k}\Gamma\left(n-k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right). \tag{B.104}$$

Finally, using our last result for the sum eq.(B.18), we can write

$$\langle\varphi_n| x |\varphi_n\rangle = -\frac{Re(\lambda)}{2\alpha}. \tag{B.105}$$

An exactly process is needed to calculate the expectation value respect the coordinate y and the result obtained

$$\langle\varphi_n| y |\varphi_n\rangle = \frac{Im(\lambda)}{2\alpha}. \tag{B.106}$$

We move on to calculate the expectation value x^2. The integration method approach follows exactly the same algebraic procedure than before, so the problem can be written as

$$\langle\varphi_n| x^2 |\varphi_n\rangle = \frac{(2\alpha)^{n+1}}{\pi n!}\sum_{k=0}^{n}\binom{n}{k}\times$$

$$\times\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} x^2\exp\left(-2\alpha\left(\left(x+\frac{Re(\lambda)}{2\alpha}\right)^2 + \left(y-\frac{Im(\lambda)}{2\alpha}\right)^2\right)\right)\left(x+\frac{Re(\lambda)}{2\alpha}\right)^{2(n-k)}\left(y-\frac{Im(\lambda)}{2\alpha}\right)^{2k} dxdy, \tag{B.107}$$

and making the same variable substitution than before eq.(B.100) and eq.(B.101) we can write

$$\langle\varphi_n| x^2 |\varphi_n\rangle = \frac{1}{\pi n!}\sum_{k=0}^{n}\binom{n}{k}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\left(\frac{\xi_1}{\sqrt{2\alpha}} - \frac{Re(\lambda)}{2\alpha}\right)^2\exp\left(-\xi_1^2 - \xi_2^2\right)\xi_1^{2(n-k)}\xi_2^{2k}\, d\xi_1\, d\xi_2. \tag{B.108}$$

Then, expanding the squared binomial factor, we have

$$\langle\varphi_n|\,x^2\,|\varphi_n\rangle = \frac{1}{\pi n!}\sum_{k=0}^{n}\binom{n}{k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\xi_1^2}{2\alpha}+\frac{Re^2(\lambda)}{(2\alpha)^2}-\frac{Re(\lambda)}{2\alpha}\frac{\xi_1}{\sqrt{2\alpha}}\right)\exp\left(-\xi_1^2-\xi_2^2\right)\xi_1^{2(n-k)}\xi_2^{2k}\,d\xi_1\,d\xi_2. \tag{B.109}$$

by the same argument did before and using the result eq.(B.103), the definition eq.(B.13) and eq.(B.18), the above expression can be written as

$$\langle\varphi_n|\,x^2\,|\varphi_n\rangle = \frac{1}{2\alpha}\frac{1}{\pi n!}\sum_{k=0}^{n}\binom{n}{k}\Gamma\left(n-k+1+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)+\frac{Re^2(\lambda)}{(2\alpha)^2}. \tag{B.110}$$

At this point, it is useful to use the next property of the gamma function, being $z\in\mathbb{Z}^+$, we have that

$$\Gamma\left(z+\frac{1}{2}\right)=\frac{(2z)!}{4^z z!}\sqrt{\pi}, \tag{B.111}$$

note that the since $n\geq k$ then the quantity $n-k+1\in\mathbb{Z}^+$ then one can get the following equality

$$\Gamma\left(n-k+1+\frac{1}{2}\right)=\left(n-k+\frac{1}{2}\right)\Gamma\left(n-k+\frac{1}{2}\right), \tag{B.112}$$

this allow us to rewrite our problem as

$$\langle\varphi_n|\,x^2\,|\varphi_n\rangle = \frac{1}{2\alpha}\left(n+\frac{1}{2}\right)-\frac{1}{2\alpha}\frac{\sum_{k=0}^{n}\binom{n}{k}k\Gamma\left(n-k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\pi n!}+\frac{Re(\lambda)^2}{4\alpha^2}, \tag{B.113}$$

we continue analyzing the expression in the middle using eq.(B.18)

$$\frac{\sum_{k=0}^{n}\binom{n}{k}k\Gamma\left(n-k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\pi n!}=\frac{\sum_{k=0}^{n}\binom{n}{k}k\Gamma\left(n-k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\sum_{k=0}^{n}\binom{n}{k}\Gamma\left(n-k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}. \tag{B.114}$$

The result of this ration is a real number but depends of the number n chosen to perform the summations. We are going to denote the result as γ_n and lets denote the

$$a_k^n=\binom{n}{k}\Gamma\left(n-k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right), \tag{B.115}$$

such that we can write

$$\frac{\sum_{k=0}^{n}ka_k^n}{\sum_{k=0}^{n}a_k^n}=\gamma_n, \tag{B.116}$$

this last expression can be rewritten as

$$\sum_{k=0}^{n}(k-\gamma_n)a_k^n=0. \tag{B.117}$$

We point out the following observation, being $m\in\mathbb{Z}$ such that $0\leq m\leq n$, and noting that

$$\binom{n}{n-m}=\frac{n!}{(n-m)!(n-(n-m))!}=\binom{n}{m} \tag{B.118}$$

this allow us to write

$$a^n_{n-m} = \binom{n}{n-m}\Gamma\left(n-(n-m)+\frac{1}{2}\right)\Gamma\left(n-m+\frac{1}{2}\right) = a^n_m. \tag{B.119}$$

We are going to illustrate some particular cases of how eq.(B.117) works that will allow us to deduce the general result of the expression. Lets make $n = 0$, then

$$\sum_{k=0}^{0}(k-\gamma_0)a^0_k = -\gamma_0 a^0_0 = 0 \tag{B.120}$$

therefore $\gamma^0 = 0$. Lets make $n = 1$, then

$$\sum_{k=0}^{1}(k-\gamma_1)a^1_k = -\gamma_1 a^1_0 + (1-\gamma_1)a^1_1 = 0 \tag{B.121}$$

using eq.(B.119), we see that $a^1_0 = a^1_1$, hence

$$(-2\gamma_1 + 1)a^1_0 = 0 \tag{B.122}$$

we note that the solution of the above expression is $\gamma_1 = 1/2$. we continue analyzing the case when $n = 2$,

$$\sum_{k=0}^{2}(k-\gamma_2)a^2_k = -\gamma_2 a^2_0 + (1-\gamma_2)a^2_1 + (2-\gamma_2)a^2_2 = 0 \tag{B.123}$$

using eq.(B.119) we see that $a^2_0 = a^2_2$, therefore

$$(-2\gamma_2 + 2)a^2_0 + (1-\gamma_2)a^2_1 = 0 \tag{B.124}$$

we note that two equations are gotten for this case, however, the equations are not linearly independent between them since one is equal to the other just by multiplying by the factor $1/2$ or by 2. This tell us that they have the same solution which is $\gamma_2 = 1$. Now, we make $n = 3$

$$\sum_{k=0}^{3}(k-\gamma_3)a^3_k = -\gamma_3 a^3_0 + (1-\gamma_3)a^3_1 + (2-\gamma_3)a^3_2 + (3-\gamma_3)a^3_3 = 0 \tag{B.125}$$

using eq.(B.119), again, we see that the following equalities holds $a^3_0 = a^3_3$ and $a^3_1 = a^3_2$, so, we have that

$$(-2\gamma_3 + 3)a^3_0 + (3-2\gamma_3)a^3_1 = 0 \tag{B.126}$$

we got the same equation in the factors which is $-2\gamma_3 + 3 = 0$, and its solution is $\gamma_3 = 3/2$.

This examples are illustrative to figure out the general proof. For any value of n given, we will have $n+1$ terms in the sum, for n odd, each one of this terms there exist another term given by eq.(B.119) such that when $k = m$ the value of the term when $k = n - m$ will add up such that

$$(m - \gamma_n)a_m^n + (n - m - \gamma_n)a_{n-m}^n = (n - 2\gamma_n)a_m^n = 0 \tag{B.127}$$

and the solution is $\gamma_n = n/2$. For the case when n is even it will happen exactly the same, however, there will exist an extra term in the middle such that $k = n/2$

$$\left(\frac{n}{2} - \gamma_n\right) a_{n/2}^n = 0 \tag{B.128}$$

however, the equations gotten here are not linearly independent since the other one its gonna be of the form eq.(B.127) and they both have the same solution which is

$$\gamma_n = n/2. \tag{B.129}$$

Finally, we conclude that

$$\frac{\sum_{k=0}^{n}\binom{n}{k}k\Gamma\left(n-k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\sum_{k=0}^{n}\binom{n}{k}\Gamma\left(n-k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)} = \frac{n}{2}, \tag{B.130}$$

and using this result we can write the expectation value can be written as

$$\langle\varphi_n|\, x^2\, |\varphi_n\rangle = \frac{1}{4\alpha}(n+1) + \frac{Re(\lambda)^2}{4\alpha^2}. \tag{B.131}$$

An exactly process can be used to prove that

$$\langle\varphi_n|\, y^2\, |\varphi_n\rangle = \frac{1}{4\alpha}(n+1) + \frac{Im(\lambda)^2}{4\alpha^2}. \tag{B.132}$$

B.6 Orthogonality of the functions

An important property to prove is the orthogonality of the functions founded, even tough if they are not ortogonal respect the degeneration index j but they are respect the energy index n. Lets calculate the inner product respect two indexes n y m, after doing the change of variables

$$\xi_1 = \sqrt{2\alpha}\left(x + \frac{Re(\lambda)}{2\alpha}\right), \quad \xi_2 = \sqrt{2\alpha}\left(y - \frac{Im(\lambda)}{2\alpha}\right), \tag{B.133}$$

where the Jacobian is $J = 1/2\alpha$ we have the following

$$\langle\varphi_n|\varphi_m\rangle = A_n A_m \exp\left(\frac{\lambda^*\lambda}{2\alpha}\right)(2\alpha)^{(n+m)/2-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi_1,\xi_2)d\xi_1 d\xi_2, \tag{B.134}$$

where the following function was defined

$$f(\xi_1,\xi_2)=\exp(-\xi_1^2-\xi_2^2)(\xi_1-i\xi_2)^n(\xi_1+i\xi_2)^m, \tag{B.135}$$

now, we are going to use polar coordinates $\xi_1=r\cos\theta$ y $\xi_2=r\sin\theta$, where the function has the form

$$f(r,\theta)=\exp(-r^2)r^{n+m}e^{i(m-n)\theta}, \tag{B.136}$$

so, we have that

$$\langle\varphi_n|\varphi_m\rangle=A_nA_m\exp\left(\frac{\lambda^*\lambda}{2\alpha}\right)(2\alpha)^{(n+m)/2-1}\int_0^\infty\int_0^{2\pi}\exp(-r^2)r^{n+m+1}e^{i(m-n)\theta}drd\theta, \tag{B.137}$$

we observe that the integral respect the angle θ vanishes for $n\neq m$ and $n,m\in\mathbb{Z}^+$ we have

$$\int_0^{2\pi}e^{i(m-n)\theta}d\theta=\frac{1}{i(m-n)}e^{i(m-n)\theta}\Big|_0^{2\pi}=0. \tag{B.138}$$

Appendix C

Calculation details for electromagnetic field and special cases

C.1 Proof that the solution for perpendicular fields satisfies the complete Schrödinger equation

In the section (2.3) we found the solution for a static and parallel electromagnetic field with the two dimensional Schrödinger's equation (2.224) to be

$$\psi_n(x,y,t) = \frac{1}{\sqrt{L_y}} \exp\left(-i\chi_n(x,y,t)\right) \phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)\right), \tag{C.1}$$

where the phase is define as follows

$$\chi_n(x,y,t) = \left(\hbar\omega_c\left(n+\frac{1}{2}\right) - \frac{1}{2m}\left(\frac{q\mathcal{E}}{\omega_c}\right)^2\right)\frac{t}{\hbar} + \left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)\left(\frac{m\omega_c}{\hbar}y - \frac{q\mathcal{E}}{\hbar\omega_c}\right). \tag{C.2}$$

Our current task is to prove that this expression satisfies the time dependent Scrödinger's equation given by eq.(2.224)

$$i\hbar\frac{\partial\psi_n}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi_n}{\partial x^2} + \frac{\partial^2\psi_n}{\partial y^2} + 2i\frac{m\omega_c}{\hbar}y\frac{\partial\psi_n}{\partial x} - \frac{m^2\omega_c^2}{\hbar^2}y^2\psi_n\right) - q\mathcal{E}y\psi_n. \tag{C.3}$$

First, we are going to write down the result of a needed set of diferentiation

$$i\hbar\frac{\partial\psi_n}{\partial t} = \left[\hbar\omega_c\left(n+\frac{1}{2}\right) + \frac{1}{2m}\left(\frac{q\mathcal{E}}{\omega_c}\right)^2 - q\mathcal{E}y\right]\psi_n + \frac{i\hbar}{\sqrt{L_y}}\exp\left(-i\chi_n\right)\frac{\partial\phi_n}{\partial t}, \tag{C.4}$$

$$2i\frac{m\omega_c}{\hbar}y\frac{\partial\psi_n}{\partial x} = 2\left(\frac{m^2\omega_c^2}{\hbar^2}y^2 - \frac{m\omega_c}{\hbar}\frac{q\mathcal{E}}{\hbar\omega_c}y\right)\psi_n + 2i\frac{m\omega_c}{\hbar}y\frac{1}{\sqrt{L_y}}\exp\left(-i\chi_n\right)\frac{\partial\phi_n}{\partial x}, \tag{C.5}$$

$$\frac{\partial^2 \psi_n}{\partial y^2} = -\frac{m^2\omega_c^2}{\hbar^2}\left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)^2 \psi_n, \tag{C.6}$$

and

$$\frac{\partial^2 \psi_n}{\partial x^2} = -\left(\frac{m\omega_c}{\hbar}y - \frac{q\mathcal{E}}{\hbar\omega_c}\right)^2 \psi_n - i\frac{2}{\sqrt{L_y}}\left(\frac{m\omega_c}{\hbar}y - \frac{q\mathcal{E}}{\hbar\omega_c}\right)\exp\left(-i\chi_n\right)\frac{\partial \phi_n}{\partial x} + \frac{1}{\sqrt{L_y}}\exp\left(-i\chi_n\right)\frac{\partial^2 \phi_n}{\partial x^2}. \tag{C.7}$$

Second, lets work with the following expression

$$\begin{aligned}\frac{\partial^2 \psi_n}{\partial x^2} + 2i\frac{m\omega_c}{\hbar}y\frac{\partial \psi_n}{\partial x} - \frac{m^2\omega_c^2}{\hbar^2}y^2\psi_n &= \left(-2\frac{m^2\omega_c^2}{\hbar^2}y^2 + 2\frac{m\omega_c}{\hbar}\frac{q\mathcal{E}}{\hbar\omega_c}y - \frac{q^2\mathcal{E}^2}{\hbar^2\omega_c^2}\right)\psi_n \\ &-i\frac{2}{\sqrt{L_y}}\left(\frac{m\omega_c}{\hbar}y - \frac{q\mathcal{E}}{\hbar\omega_c}\right)\exp\left(-i\chi_n\right)\frac{\partial \phi_n}{\partial x} + \frac{1}{\sqrt{L_y}}\exp\left(-i\chi_n\right)\frac{\partial^2 \phi_n}{\partial x^2} \\ &+2\left(\frac{m^2\omega_c^2}{\hbar^2}y^2 - \frac{m\omega_c}{\hbar}\frac{q\mathcal{E}}{\hbar\omega_c}y\right)\psi_n + 2i\frac{m\omega_c}{\hbar}y\frac{1}{\sqrt{L_y}}\exp\left(-i\chi_n\right)\frac{\partial \phi_n}{\partial x},\end{aligned} \tag{C.8}$$

doing some simplification one can write

$$\frac{\partial^2 \psi_n}{\partial x^2} + 2i\frac{m\omega_c}{\hbar}y\frac{\partial \psi_n}{\partial x} - \frac{m^2\omega_c^2}{\hbar^2}y^2\psi_n = -\frac{q^2\mathcal{E}^2}{\hbar^2\omega_c^2}\psi_n + i\frac{2}{\sqrt{L_y}}\left(\frac{q\mathcal{E}}{\hbar\omega_c}\right)\exp\left(-i\chi_n\right)\frac{\partial \phi_n}{\partial x} + \frac{1}{\sqrt{L_y}}\exp\left(-i\chi_n\right)\frac{\partial^2 \phi_n}{\partial x^2}, \tag{C.9}$$

thus, substituting this result, eq.(C.4), eq.(C.6) in eq.(2.224) and multiplying by $\sqrt{L_y}\exp\left(i\chi_n\right)$, we can write the next equality

$$\begin{aligned}\left[\hbar\omega_c\left(n + \frac{1}{2}\right) - \frac{1}{2m}\left(\frac{q\mathcal{E}}{\omega_c}\right)^2 - q\mathcal{E}y\right]\phi_n + i\hbar\frac{\partial \phi_n}{\partial t} = \\ -\frac{\hbar^2}{2m}\left(-\frac{q^2\mathcal{E}^2}{\hbar^2\omega_c^2}\phi_n + 2i\left(\frac{q\mathcal{E}}{\hbar\omega_c}\right)\frac{\partial \phi_n}{\partial x} + \frac{\partial^2 \phi_n}{\partial x^2} - \frac{m^2\omega_c^2}{\hbar^2}\left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)^2\phi_n\right) - q\mathcal{E}y\phi_n,\end{aligned} \tag{C.10}$$

we continue doing some algebraic simplification to get the following

$$\left[\hbar\omega_c\left(n + \frac{1}{2}\right)\right]\phi_n + i\hbar\frac{\partial \phi_n}{\partial t} = -\frac{\hbar^2}{2m}\left(2i\left(\frac{q\mathcal{E}}{\hbar\omega_c}\right)\frac{\partial \phi_n}{\partial x} + \frac{\partial^2 \phi_n}{\partial x^2} - \frac{m^2\omega_c^2}{\hbar^2}\left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)^2\phi_n\right), \tag{C.11}$$

note that making the change of variable

$$\phi_n = \phi_n \left(\sqrt{\frac{m\omega_c}{\hbar}} \left(x - \frac{q\mathcal{E}}{m\omega_c} t \right) \right), \qquad \xi = \sqrt{\frac{m\omega_c}{\hbar}} \left(x - \frac{q\mathcal{E}}{m\omega_c} t \right) \tag{C.12}$$

the next equality can be written

$$\frac{\partial \phi_n}{\partial t} = -\frac{q\mathcal{E}}{m\omega_c} \frac{\partial \phi_n}{\partial x} \tag{C.13}$$

hence, we can use it to simplify the prior result and write

$$\left[\hbar\omega_c \left(n + \frac{1}{2} \right) \right] \phi_n = -\frac{\hbar^2}{2m} \left(\frac{\partial^2 \phi_n}{\partial x^2} - \frac{m^2\omega_c^2}{\hbar^2} \left(x - \frac{q\mathcal{E}}{m\omega_c} t \right)^2 \phi_n \right), \tag{C.14}$$

Finally, one can note that what is left is nothing but the harmonic oscillator equation with $\xi = \sqrt{\frac{m\omega_c}{\hbar}} \left(x - \frac{q\mathcal{E}}{m\omega_c} t \right)$ and since ϕ_n is the solution of the harmonic oscillator we conclude that the above equation holds as true and therefore eq.(C.1) satisfies the time dependent Schrödinger equation eq.(2.224).

C.2 Charged particle with electromagnetic field (parallel case)

Lets analyze the situation where the electric and magnetic field directions are arranged in a way that they are parallel between them but still constant. A posible selection of the vector fields are $\mathbf{B} = (0, B, 0)$ and $\mathbf{E} = (0, \mathcal{E}, 0)$ that can be described by the following potentials

$$\mathbf{A} = B(z, 0, 0), \tag{C.15}$$

and

$$\phi = -\mathcal{E}y, \tag{C.16}$$

this defines the following Schrödinger's equation

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m} \left[(\hat{p}_x - m\omega_c z)^2 \psi + \hat{p}_y^2 \psi + \hat{p}_z^2 \psi \right] - q\mathcal{E}y\psi. \tag{C.17}$$

expanding the binomial squared and substituting the momentum operators, we get the following expression

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} - 2i \frac{m\omega_c}{\hbar} z \frac{\partial \psi}{\partial x} - \frac{m^2\omega_c^2}{\hbar^2} z^2 \psi \right] - q\mathcal{E}y\psi. \tag{C.18}$$

Similarly to the prior cases, we can take the Fourier transform respect the x variable, so, we can write down the following equation in the Fourier space (k, y, z, t), defining

$$\bar{\psi} = \mathcal{F}_x\{\psi\}, \tag{C.19}$$

we can write

$$i\hbar \frac{\partial \bar{\psi}}{\partial t} = -\frac{\hbar^2}{2m} \left[-k^2 \bar{\psi} + \frac{\partial^2 \bar{\psi}}{\partial y^2} + \frac{\partial^2 \bar{\psi}}{\partial z^2} - 2 \frac{m\omega_c}{\hbar} z k \bar{\psi} - \frac{m^2\omega_c^2}{\hbar^2} z^2 \bar{\psi} \right] - q\mathcal{E}y\bar{\psi}, \tag{C.20}$$

or completing the squared quantity,

$$i\hbar\frac{\partial\bar{\psi}}{\partial t} = -\frac{\hbar^2}{2m}\left[\frac{\partial^2\bar{\psi}}{\partial y^2} + \frac{\partial^2\bar{\psi}}{\partial z^2} - \left(\frac{m\omega_c}{\hbar}z + k\right)^2\bar{\psi}\right] - q\mathcal{E}y\bar{\psi}. \tag{C.21}$$

From this expression, it can be seen that it is separable, proposing the following $\bar{\psi} = \varphi_1(y,t)\varphi_2(k,z,t)$

$$i\hbar\frac{1}{\varphi_1}\frac{\partial\varphi_1}{\partial t} + i\hbar\frac{1}{\varphi_2}\frac{\partial\varphi_2}{\partial t} = -\frac{\hbar^2}{2m}\frac{1}{\varphi_1}\frac{\partial^2\varphi_1}{\partial y^2} - q\mathcal{E}y - \frac{\hbar^2}{2m}\left[\frac{1}{\varphi_2}\frac{\partial^2\varphi_2}{\partial z^2} - \left(\frac{m\omega_c}{\hbar}z + k\right)^2\right]. \tag{C.22}$$

then, we have the following couple of ordinary differential equations

$$i\hbar\frac{\partial\varphi_1}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\varphi_1}{\partial y^2} - q\mathcal{E}y\varphi_1, \tag{C.23}$$

$$i\hbar\frac{\partial\varphi_2}{\partial t} = -\frac{\hbar^2}{2m}\left[\frac{\partial^2\varphi_2}{\partial z^2} - \left(\frac{m\omega_c}{\hbar}z + k\right)^2\varphi_2\right]. \tag{C.24}$$

It can be seen immediately that eq.(C.24) is the displaced harmonic oscillator and has the following solution

$$g_n = \varphi_2(k,z,t) = e^{-i\frac{E_n}{\hbar}t}\phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left(z + \frac{\hbar}{m\omega_c}k\right)\right), \tag{C.25}$$

where the harmonic oscillator was define as

$$\phi_n(\xi) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega_c}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\xi^2}{2}\right)H_n(\xi), \tag{C.26}$$

and the eigenvalues are

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right). \tag{C.27}$$

On the other hand, equation (C.23) is the quantum bouncer and the solution can be found in reference [84, p. 109] and can be written as

$$\varphi_1(y,t) = \frac{A}{\sqrt{t}}\int dy'\varphi_0(y')\exp\left(\frac{im}{2\hbar t}\left(y - y' + \frac{q\mathcal{E}t^2}{2m}\right)^2 - i\frac{(q\mathcal{E})^2t^3}{6m\hbar} + iq\mathcal{E}y'\frac{t}{\hbar}\right), \tag{C.28}$$

where $\varphi_0(y')$ is an initial state of the particle. Note that the above expression is a solution for the complete Schrödinger equation define in (C.23). Therefore the solution in the Fourier space is

$$\bar{\psi}(k,y,z,t) = \varphi_1(y,t)e^{-i\frac{E_n}{\hbar}t}\phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left(z + \frac{\hbar}{m\omega_c}k\right)\right), \tag{C.29}$$

then, we can get the solution in the original space by performing the inverse Fourier transform,

$$\psi(x,y,z,t) = \mathcal{F}_x^{-1}\{\bar{\psi}\} = \varphi_1(y,t)e^{-i\frac{E_n}{\hbar}t}\int\limits_{\mathbb{R}} e^{-ikx}\phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left(z + \frac{\hbar}{m\omega_c}k\right)\right)dk, \tag{C.30}$$

making the following change of variable

$$k = \sqrt{\frac{m\omega_c}{\hbar}}\xi - \frac{m\omega_c}{\hbar}z, \tag{C.31}$$

and using the result that of the appendix (A.1) we get

$$\psi_n(x,y,z,t) = A\varphi_1(y,t)e^{-i\frac{E_n}{\hbar}t}e^{i\frac{m\omega_c}{\hbar}xz}\phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}x\right), \tag{C.32}$$

where A is the normalization constant.

Appendix D

Electromagnetic flux quantization and density of states

In this appendix we discuss different approaches that show the possible reasons for the quantization of both field, the magnetic and electric fields. We begin analyzing the solution obtained for the partial differential equation using solely the Landau's gauge and extend the discussion to the case where the electric field is added.

D.1 Density of states and magnetic flux quantization

Classically, the particle is expected to describe a circular trajectory on the plane, therefore, the radius of this trajectory can be written as $r^2 = x^2 + y^2$, therefore, the area described for this trajectory is $A = \pi r^2$. Our task here is to calculate the expectation value of this area, that is

$$\langle A \rangle = \pi \langle x^2 \rangle + \pi \langle y^2 \rangle \,. \tag{D.1}$$

We begin the calculations using the solutions found in chapter (2) for the Landau's gauge, eq.(2.49),

$$\psi_n(x,y,z) = \frac{1}{\sqrt{L_y L_z}} \exp\left(-i\frac{m\omega_c}{\hbar}xy + i\frac{\sqrt{2mE_2}}{\hbar}z\right)\phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}x\right), \tag{D.2}$$

limiting the analysis for the two dimensional system we have

$$\psi_n(x,y) = \frac{1}{\sqrt{L_y}} \exp\left(-i\frac{m\omega_c}{\hbar}xy\right)\phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}x\right), \tag{D.3}$$

then, one can use the properties of the harmonic oscillator to calculate the following integral

$$\langle \psi_n | \, x^2 \, | \psi_n \rangle = \frac{1}{L_y} \int\limits_{-L_y}^{L_y} \int\limits_{-\infty}^{\infty} \phi_n^*\left(\sqrt{\frac{m\omega_c}{\hbar}}x\right) x^2 \phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}x\right) dx\, dy \tag{D.4}$$

making the change of variable

$$\xi = \sqrt{\frac{m\omega_c}{\hbar}}x, \tag{D.5}$$

and solving the integral respect y we have that

$$\langle\psi_n| x^2 |\psi_n\rangle = 2\frac{\hbar}{m\omega_c}\sqrt{\frac{\hbar}{m\omega_c}}\int_{-\infty}^{\infty}\phi_n^*(\xi)\xi^2\phi_n(\xi)\, d\xi \tag{D.6}$$

now, we can use the following recurrence relation of the Hermite polynomials

$$\xi H_n(\xi) = nH_{n-1}(\xi) + \frac{1}{2}H_{n+1}(\xi), \tag{D.7}$$

to write

$$\xi^2\phi_n(\xi) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega_c}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\xi^2}{2}\right)\left(n(n-1)H_{n-2}(\xi) + \left(n+\frac{1}{2}\right)H_n(\xi) + \frac{1}{4}H_{n+2}(\xi)\right), \tag{D.8}$$

using the fact that the Hermite polynomials are ortogonal respect a gaussian function, that is

$$\int_{-\infty}^{\infty} e^{-\xi^2}H_nH_m(\xi)d\xi = 2^n n!\sqrt{\pi}\delta_{n,m} \tag{D.9}$$

we can write the following expression

$$\langle\psi_n| x^2 |\psi_n\rangle = 2\frac{\hbar}{m\omega_c}\left(n+\frac{1}{2}\right). \tag{D.10}$$

Continuing, the expectation value respect y^2 is really straight forward and the following result is obtained

$$\langle\psi_n| y^2 |\psi_n\rangle = 2\frac{L_y^2}{3}. \tag{D.11}$$

Therefore, the expected value of the area described by the particles trajectory is

$$\langle\psi_n| A |\psi_n\rangle = 2\pi\frac{\hbar}{m\omega_c}\left(n+\frac{1}{2}\right) + 2\pi\frac{L_y^2}{3}, \tag{D.12}$$

hence, the area described between two energy levels n_1 and n_2 is

$$\Delta A = \langle\psi_{n_2}| A |\psi_{n_2}\rangle - \langle\psi_{n_1}| A |\psi_{n_1}\rangle = 2\pi\frac{\hbar}{m\omega_c}(n_2 - n_1). \tag{D.13}$$

Finally, we see that the density of states per unit area can be written as

$$\frac{m\omega_c}{\hbar} = 2\pi\frac{\Delta n}{\Delta A}, \tag{D.14}$$

where $\Delta n = n_2 - n_1$.

On the other hand, this calculation can be done for the symmetric gauge eigenfunctions, the calculation of the expected values of the coordinates squared is more complicated but can be found in the appendix (B.5). The results are the following

$$\langle\varphi_n| x^2 |\varphi_n\rangle = \frac{1}{4\alpha}(n+1) + \frac{Re(\lambda)^2}{4\alpha^2}, \tag{D.15}$$

and

$$\langle\varphi_n| y^2 |\varphi_n\rangle = \frac{1}{4\alpha}(n+1) + \frac{Im(\lambda)^2}{4\alpha^2}, \tag{D.16}$$

where $\alpha = m\omega_c/4\hbar$. Therefore, the expectation value of the area described by the particle using this gauge is

$$\langle\varphi_n| A |\varphi_n\rangle = \frac{\pi}{2\alpha}(n+1) + \pi\frac{\lambda^*\lambda}{4\alpha^2}, \tag{D.17}$$

and the are described between two energy levels n_2 and n_1 is

$$\Delta A = 2\pi\frac{\hbar}{m\omega_c}(n_2 - n_1). \tag{D.18}$$

and the density of states per unit area is exactly the same than in the prior case and is given by the expression eq.(D.14).

Alternatively, the solution eq.(D.3) has an interesting property, if one propose invariance over the following boundary condition at $x = L_x$ and $y \to y + L_y$

$$\psi_n(L_x, y + L_y) = e^{-i\frac{m\omega_c}{\hbar}L_xL_y}\psi_n(L_x, y) = \psi_n(L_x, y), \tag{D.19}$$

one has that the quantity

$$\frac{m\omega_c}{\hbar}L_xL_y = 2\pi k, \quad k \in \mathbb{Z}. \tag{D.20}$$

Even though, it looks like the same result that the density of states, the physical meaning is not the same, since the index k is an integer related to the periodicity and is not related to the difference of two energy levels.

D.2 Electromagnetic flux quantization due to symmetry invariance

In the first attempt to study the quantum dynamics of a charged particle under a static magnetic field Landau did the remarkable conclusion that the density of states per unit area must be proportional to the magnetic field intensity, implying that the magnetic flux must be quantized [1]. This result has been useful to prove the resistance quantization in the integer quantum Hall effect discovered by Klitzing [3]. This phenomena has been explained by Laughlin as a consequence of the Schrödinger equation invariance under unitary gauge transformations [4]. Later on, the fractional quantum Hall effect was discovered experimentally [22] by Stormer but explained by Laughlin where he used a fractional charge quasiparticle hypothesis to do so [43,44]. However, a generalization of Laughlin's invariance argument has been done by Tao and Wu [52]. In this last study, the authors made the remarkable conclusion that the fractional quantum Hall effect exist only if the system's ground state is degenerated, which is, in fact, a contradiction to Laughlin uniqueness ground state argument show in reference [44]. However, in this section we show that the conclusion of Tao and Wu were in the right direction, since the operators responsable of the system degeneration defines unitary transformations that are responsables of the quantization of the resistivity and generates the fractional quantum Hall effect.

The no relativistic Hamiltonian that describes the quantum dynamics of a charge particle under an electromagnetic field is written as

$$\hat{\mathcal{H}} = \frac{1}{2m}\left(\hat{P} - \frac{q}{c}A\right)^2 + q\phi, \tag{D.21}$$

where q is the particle charge, m is the particle mass, c is the speed of light, $\mathbf{A}$ is the magnetic vector potential such that the magnetic field is given by its curl, $\mathbf{B} = \nabla \times B$, $\phi = \phi(x, y, z)$ is the electric potential such that the electric field is given by $\mathbf{E} = -\nabla\phi$ and $\hat{P} = -i\hbar\nabla$ is the momentum operator. Along this work, we study the cases where the magnetic field is described by the Landau's gauge and the symmetric gauge such that the magnetic field is constant and parallel to the z axis, hence, the vector potential is written as $\mathbf{A} = (A_x(x, y), A_y(x, y), 0)$, this allow us to write the Hamiltonian as $\hat{\mathcal{H}} = \mathbf{H} + \hat{p}_z^2/2m$ where the two dimensional Hamiltonian was define as

$$\mathbf{H} = \frac{1}{2m}\left(\left(\hat{p}_x - \frac{q}{c}A_x\right)^2 + \left(\hat{p}_y - \frac{q}{c}A_y\right)^2\right) + q\phi. \tag{D.22}$$

Since the coordinates of the vector $\mathbf{A}$ do not depend on z, then this coordinate can be separated and the solutions has the form of $\Psi(x, y, z) \sim \psi(x, y)e^{i\frac{\sqrt{2mE_z}}{\hbar}z}$ where $\psi = \psi(x, y)$ is the solution of the Schrödinger equation defined by the Hamiltonian eq.(D.22). From now on, for the sake of simplicity, we are going to focus on the two dimensional system described by the Hamiltonian $\mathbf{H}$.

D.2.1 Symmetries for the Landau's gauge

The two dimensional Hamiltonian system without electric field $\mathbf{E} = 0$ but with constant magnetic field described by the Landau gauge $\mathbf{A} = B(-y, 0, 0)$ is written as

$$\mathbf{H} = \frac{1}{2m}\left((\hat{p}_x + m\omega_c y)^2 + \hat{p}_y^2\right), \tag{D.23}$$

were $\omega_c = qB/mc$, was define. It has been proved that the canonic momentum operators described by the inversion of the magnetic field direction and using the alternative Landau's gauge are conserved [85], those operators are define as

$$\hat{\pi}'_x = \hat{p}_x, \tag{D.24}$$

and

$$\hat{\pi}'_y = \hat{p}_y + m\omega_c x, \tag{D.25}$$

such that

$$[\hat{\pi}'_x, \mathbf{H}] = [\hat{\pi}'_y, \mathbf{H}] = 0. \tag{D.26}$$

The non separable eigenfunctions are [86, 87]

$$f_n^0 = \frac{1}{\sqrt{L_y}}e^{-i\frac{m\omega_c}{\hbar}xy}\phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}x\right), \tag{D.27}$$

where the harmonic oscillator function was defined as

$$\phi_n(\chi) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega_c}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\chi^2}{2}\right)H_n(\chi), \tag{D.28}$$

having the Landau's levels as eigenvalues

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right) \quad n \in \mathbb{Z}^+. \tag{D.29}$$

It has been shown that this system is numerable degenerated since each time we apply the operator eq.(D.24) on the eigenfunction eq.(D.27) it will give us back new eigenfunction [85, 88], this eigenfunctions are denoted as

$$(\hat{\pi}'_x)^j f_n^0 = f_n^j, \quad j \in \mathbb{Z}^+, \tag{D.30}$$

each one of this functions satisfies the eigenvalue equation

$$\mathbf{H}f_n^j = E_n f_n^j. \tag{D.31}$$

On the other hand, its not difficult to prove that the operator eq.(D.25) does not give any new eigenfunction since $\hat{\pi}'_y\psi_n = 0$ but, in fact, it can be used to build up a second eigenvalue equation regarding the index j

$$\hat{\pi}'_y\hat{\pi}'_x f_n^j = im\omega_c\hbar(j+1)f_n^j. \tag{D.32}$$

Now, the conserved operators are generators of transformations and can be used to define conserved unitary operators,

$$\hat{U}_x = e^{-i\frac{\delta x}{\hbar}\hat{\pi}'_x}, \tag{D.33}$$

and

$$\hat{U}_y = e^{-i\frac{\delta y}{\hbar}\hat{\pi}'_y}, \tag{D.34}$$

where δx and δy are displacement, note that the operator eq.(D.33) is the translation operators along the x axis and the operator eq.(D.34) is the magnetic translational operator with inversed magnetic field. Both operators leaves the Hamiltonian invariant, therefore, it has the following symmetries

$$\mathbf{H} = \hat{U}_x\mathbf{H}\hat{U}_x^\dagger, \quad \mathbf{H} = \hat{U}_y\mathbf{H}\hat{U}_y^\dagger. \tag{D.35}$$

However, in Schrödinger scheme, the general solution for this situation can be written as

$$\Psi(x,y,t) = \sum_{n,j} C_{n,j}e^{-i\frac{E_n}{\hbar}t}f_n^j(x,y). \tag{D.36}$$

In general, is highly difficult to work with the whole set of eigenfunctions but there is a specific choice of constant which simplifies the situation, note that if we chose the constants to be

$$C_{n,j} = C_n\frac{1}{j!}\frac{\delta x^j}{(i\hbar)^j}, \tag{D.37}$$

the general solution has the following form

$$\Psi(x,y,t) = \sum_n C_n e^{-i\frac{E_n}{\hbar}t}\hat{U}_x f_n^0(x,y) = \sum_n C_n e^{-i\frac{E_n}{\hbar}t}f_n^0(x-\delta x, y). \tag{D.38}$$

Hence, it can be seen that the effect of the degeneration in the system is to displace it along the x axis by an amount of δx. Now, the action of the operator eq.(D.34) on the above expression gives

$$\hat{U}_y\Psi(x,y) = e^{-i\frac{m\omega_c}{\hbar}\delta x\delta y}\Psi(x,y), \tag{D.39}$$

therefore it remains invariant if the phase is such that

$$\frac{m\omega_c}{\hbar}\delta x\delta y = 2\pi l, \quad l\in\mathbb{Z}. \tag{D.40}$$

This is the same quantization condition that Landau presented for the density of states, however, in this case the index l has no relation with the states index n. Note that the above condition implies the quantization of the magnetic flux.

D.2.2 Symmetries obtained with the symmetric gauge

A magnetic field in the same direction and with the same magnitud than in the previous case can be described using the so called symmetric gauge, which is written as follows

$$\mathbf{A} = \frac{B}{2}(-y, x, 0). \tag{D.41}$$

This gauge selection give us the two dimensional Hamiltonian eq.(D.22)

$$\mathbf{H} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 - m\omega_c \hat{L}_z + \frac{m^2\omega_c^2}{4}(x^2+y^2)\right). \tag{D.42}$$

where the angular momentum was define $\hat{L}_z = x\hat{p}_y - y\hat{p}_x$. The non separable solution of this expression is [86, 87]

$$\varphi_n(x,y) = \frac{e^{-|\lambda|^2/4\alpha}}{\sqrt{(2\alpha)^{n-1}\pi n!}} \exp\left(-\alpha(x^2+y^2) - \lambda(x+iy)\right)\left(2\alpha(x-iy)+\lambda\right)^n, \quad \alpha = \frac{m\omega_c}{4\hbar}, \tag{D.43}$$

where $\lambda \in \mathbb{C}$ is an integration constant and the eigenvalues are its same Landau's levels defined in eq.(D.29). Similarly than the previous case, the conserved operators are the canonic momentum defined by the inversion of the magnetic field direction [85]

$$\widetilde{\pi}_x = \hat{p}_x - \frac{m\omega_c}{2}y, \qquad \widetilde{\pi}_y = \hat{p}_y + \frac{m\omega_c}{2}x, \tag{D.44}$$

and even though if the angular momentum $\hat{L}_z$ is also conserved, it turns out that it can be written as a linear combination of this two operators. Hence, we define the notation $\varphi_n(x,y) = f_n^0(x,y) = g_n^0(x,y)$, and the set of eigenfunctions can be obtained by the application of the operators eq.(D.44) on the function eq.(D.43) such that $(\hat{\pi}'_x)^j\varphi_n(x,y) = f_n^j(x,y)$ and $(\hat{\pi}'_y)^j\varphi_n(x,y) = g_n^j(x,y)$ for $j = 0, 1, 2, ...$ All this expressions satisfies the eigenvalue equation respect the energy level index n

$$\hat{\mathcal{H}} f_n^{j+1} = E_n f_n^{j+1}, \qquad \hat{\mathcal{H}} g_n^{j+1} = E_n g_n^{j+1}, \tag{D.45}$$

besides, a second eigenvalue equation regarding the degeneration index j [85]

$$(\widetilde{\pi}_y + i\widetilde{\pi}_x + 2\hbar\lambda)\widetilde{\pi}'_x f_n^j = i\hbar m\omega_c(j+1)f_n^j, \qquad (i\widetilde{\pi}_y - \widetilde{\pi}_x + 2i\hbar\lambda)\widetilde{\pi}_y g_n^j = i\hbar m\omega_c(j+1)g_n^j. \tag{D.46}$$

Hence, we can define unitary transforms using the operators of this second eigenvalue expressions as follows

$$\hat{U}_x = e^{-i\frac{\delta x}{\hbar}\widetilde{\pi}_x}, \tag{D.47}$$

and

$$\hat{U}_y = \exp\left(-i\frac{\delta y}{\hbar}\left(\widetilde{\pi}_y + i\widetilde{\pi}_x + 2\hbar\lambda\right)\right), \tag{D.48}$$

Again, note that those operators are translation operators times a phase factor. This two unitary operators define us the symmetries of the Hamiltonian as

$$\mathbf{H} = \hat{U}_x\mathbf{H}\hat{U}_x^\dagger, \quad \mathbf{H} = \hat{U}_y\mathbf{H}\hat{U}_y^\dagger. \tag{D.49}$$

Similarly to the previous case, the general solution can be written as a linear combination of the degenereted eigenfunctions, that is,

$$\Psi(x,y) = \sum_{n,j} C_{n,j} f_n^j(x,y), \tag{D.50}$$

if we define the same constants as shown in eq.(D.37), the general solution can be written as

$$\Psi(x,y) = \sum_n C_n \hat{U}_x \varphi_n(x,y) = \exp\left(i\frac{m\omega_c}{2\hbar} y\delta x\right) \sum_n C_n \varphi_n(x-\delta x, y). \tag{D.51}$$

Now, applying the operator eq.(D.48) on this expression, is possible to show that the next equality is obtained

$$\hat{U}_y \Psi(x,y) = \exp\left(-i\frac{m\omega_c}{\hbar}\delta x\delta y\right)\Psi(x,y), \tag{D.52}$$

hence, the invariance of the system with this symmetry requires the quantization of the magnetic flux

$$\frac{m\omega_c}{\hbar}\delta x\delta y = 2\pi l, \quad l \in \mathbb{Z}. \tag{D.53}$$

A similar process can be follow using the functions g_n^j and the same result is obtained.

D.2.3 Electromagnetic flux quantization

Here, we are going to analyze the system symmetries when a static electric field is added such that it is perpendicular to the magnetic field. This case is of particular interest since this set up is widely used in quantum Hall effect experiments [3, 22]. To describe this system we are going to use the Landau's gauge used in section ($D.2.1$), $\mathbf{A} = B(-y, 0, 0)$, and the electrical potential is chosen to be $\phi = -\mathcal{E}y$, where $\mathcal{E}$ is the electric field intensity. Note that this potentials selection satisfies the condition that $\mathbf{B}\cdot\mathbf{E} = 0$. The two dimensional Hamiltonian described by eq.(D.22) is

$$\mathbf{H} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 + 2m\omega_c y\hat{p}_x + m^2\omega_c^2 y^2\right) - q\mathcal{E}y, \tag{D.54}$$

and the non separable variable solution of the complete Schrödinger's equation

$$i\hbar\frac{\partial\psi}{\partial t} = \mathbf{H}\psi, \tag{D.55}$$

is [86, 87]

$$\psi_n(x,y,t) = \frac{1}{\sqrt{L_y}}\exp\left(-i\chi_n(x,y,t)\right)\phi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)\right), \tag{D.56}$$

where $\phi_n(\xi)$ is the harmonic oscillator defined in eq.(D.28) and the phase has been defined as follows

$$\chi_n(x,y,t) = \left(\hbar\omega_c\left(n+\frac{1}{2}\right) - \frac{1}{2m}\left(\frac{q\mathcal{E}}{\omega_c}\right)^2\right)\frac{t}{\hbar} + \left(x - \frac{q\mathcal{E}}{m\omega_c}t\right)\left(\frac{m\omega_c}{\hbar}y - \frac{q\mathcal{E}}{\hbar\omega_c}\right). \tag{D.57}$$

Similarly as we did in section ($D.2.1$), we define the two canonic momentum operator due to the inversion of the magnetic fields direction as in expressions eq.(D.24) and eq.(D.25), this help us to define the following set of operators

$$\hat{\pi}'_x = \hat{p}_x, \tag{D.58}$$

$$\hat{\Pi}'_y = \hat{\pi}'_y - q\mathcal{E}t, \tag{D.59}$$

$$\hat{E} = i\hbar\frac{\partial}{\partial t}, \tag{D.60}$$

and using the Heisenberg definition for an evolution of an operator $\hat{f}$

$$\frac{d\hat{f}}{dt} = \frac{1}{i\hbar}[\hat{f}, \mathbf{H}] + \frac{\partial\hat{f}}{\partial t}, \tag{D.61}$$

it can be found that all the three operators eq.(D.58), eq.(D.59) and eq.(D.60) are conserved. Since $\hat{\Pi}'_y\psi_n = 0$ the only two operators that define new solutions of the complete Schrödinger equation eq.(D.55) are eq.(D.58) and eq.(D.60). Using the operator eq.(D.58) we can define the set of functions $f_n^j(x,y,t) = \hat{p}_x^j\psi_n(x,y,t)$ where $n, j = 0, 1, 2, \ldots$. Each one of this functions satisfies the time dependent Schrödinger equation, that is,

$$\mathbf{H} f_n^j = i\hbar\frac{\partial f_n^j}{\partial t}, \tag{D.62}$$

besides, the operator eq.(D.59) can be used to build an eigenvalue equation regarding the degeneration index j,

$$\hat{\Pi}'_y\hat{\pi}'_x f_n^j = im\omega_c\hbar(j+1)f_n^j. \tag{D.63}$$

On the other hand, the energy operator eq.(D.60) do not share bases with the Hamiltonian, instead it is a generator of solutions of the complete Schrödinger equation. This is easy to prove, using the fact that any power of this operator commutes with the Hamiltonian, that is, $[(\hat{E})^j, \hat{H}] = 0$, where $j \in \mathbb{Z}^+$, then it follows that

$$(\hat{E})^j\hat{H}\psi_n = \hat{H}(\hat{E})^j\psi_n, \tag{D.64}$$

defining the functions $g_n^j(x,y,t) = (\hat{E})^j\psi_n(x,y,t)$ and using the expression eq.(D.55), it follows that

$$i\hbar\frac{\partial g_n^j}{\partial t} = \hat{H}g_n^j. \tag{D.65}$$

Hence, the energy operator combined with the operator eq.(D.59) can be used to find a second eigenvalue expression

$$\hat{\Pi}'_y\hat{E}g_n^j = iq\mathcal{E}\hbar(j+1)g_n^j. \tag{D.66}$$

An interesting observation about the expressions eq.(D.63) and eq.(D.66) is that the eigenvalues of the first one are proportional to the magnetic field intensity (since $\omega_c = qB/mc$) and the eigenvalues of the second one are proportional to the electric field intensity $\mathcal{E}$. At this point, is posible to define the unitary operators

$$\hat{U}_x = \exp\left(-\frac{i}{\hbar}\delta x\hat{\pi}'_x\right), \tag{D.67}$$

$$\hat{U}_y = \exp\left(-\frac{i}{\hbar}\delta y\hat{\Pi}'_y\right), \tag{D.68}$$

and

$$\hat{U}_t = \exp\left(i\frac{\delta t}{\hbar}\hat{E}\right). \tag{D.69}$$

Secondly, one can rewrite the equation eq.(D.55) as $(\mathbf{H} - \hat{E})\psi = 0$ and define the operator $\hat{O} = \mathbf{H} - \hat{E}$ which defines the complete Schrödinger equation. In this case, since the operator eq.(D.59) depends explicitly on the time variable t, the symmetries are defined regard the operator $\hat{O}$ instead of the Hamiltonian alone. Therefore, the complete Schrödinger equation has the following symmetries

$$\hat{O} = \hat{U}_x\hat{O}\hat{U}_x^\dagger, \quad \hat{O} = \hat{U}_y\hat{O}\hat{U}_y^\dagger, \quad \hat{O} = \hat{U}_t\hat{O}\hat{U}_t^\dagger. \tag{D.70}$$

Then, the general solution to this problem is written as

$$\Psi(x,y,t) = \sum_{n,j,j'} C_{n,j,j'}(\hat{E})^{j'}\hat{p}_x^j\psi_n, \tag{D.71}$$

choosing the following form of the constants

$$C_{n,j,j'} = C_n \frac{1}{j!} \frac{\delta x^j}{(i\hbar)^j} \frac{1}{j'!} \frac{\delta t^{j'}}{(-i\hbar)^{j'}}, \tag{D.72}$$

the general solution can be written as

$$\Psi(x, y, t) = \sum_n C_n \hat{U}_t \hat{U}_x \psi_n(x, y, t) = \sum_n C_n \psi_n(x - \delta x, y, t - \delta t), \tag{D.73}$$

applying the operator eq.(D.59) on this expression one has that

$$\hat{U}_y \Psi(x, y, t) = \exp\left(-i\frac{m\omega_c}{\hbar}\delta x \delta y\right) \exp\left(i\frac{q\mathcal{E}}{\hbar}\delta t \delta y\right) \Psi(x, y, t), \tag{D.74}$$

and we see that this expression remains invariant only if the following quantities are quantized (note that the quantities δx and δy are not infinitesimal)

$$\frac{m\omega_c}{\hbar}\delta x \delta y = 2\pi l, \quad l \in \mathbb{Z}, \tag{D.75}$$

and

$$\frac{q\mathcal{E}}{\hbar}\delta t \delta y = 2\pi k, \quad k \in \mathbb{Z}. \tag{D.76}$$

The first one of the above quantities is the usual quantization of the magnetic flux which implies the quantization of the magnetic flux. The second one implies the quantization of the electric flux, however, it can be rewritten as

$$\mathcal{E}\delta y \frac{\delta t}{q} = \frac{h}{q^2} k \tag{D.77}$$

where $h = 2\pi\hbar$, note that $\mathcal{E}\delta y$ is the voltage generated between two points along the y axis and $q/\delta t$ is the current generated by a single particle, this tell us that the above equality can be regarded as a resistance quantization proportional to the von Klitzing constant h/q^2.

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